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NUMBER THEORY

Terrill, H. M., and Sweeny, Lucile. Two constants connected with the theory of prime numbers. J. Franklin Inst. 239, 242–243 (1945). [MF 12062]

The authors determine to 23 decimal places the value of the limit

$$\lim_{x\to\infty} \left(\sum_{p\leq x} p^{-1}-\log\log x\right)=C+\sum_{p} \{p^{-1}+\log(1-p^{-1})\}$$

$$=.26149\ 72128\ 47642\ 78375\ 543,$$

where C is Euler's constant [see, for example, Hardy and Wright, Theory of Numbers, Oxford University Press, 1938, p. 350]. Two calculations were made of the last sum, first writing it as $-\sum_{k=2}^{\infty}k^{-1}\sum_p p^{-k}$ and then as $-\sum_{k=2}^{\infty}\mu(k)\zeta(k)$. Values of $\sum p^{-k}$ and $\zeta(k)$ were taken from H. T. Davis, Tables of the Higher Mathematical Functions, The Principia Press, Bloomington, Ind., 1935, v. 2, pp. 244, 249, 250.

D. H. Lehmer (Berkeley, Calif.).

Benckert, Curt Ragnar. A variant of the sieve of Eratosthenes. Ark. Mat. Astr. Fys. 29B, no. 13, 5 pp. (1943). (Swedish) [MF 12024]

Write an infinite multiplication table in the form of a matrix with elements $a_{ik}=(i-1)(k-1)$. This matrix contains all positive integers except primes. In similar ways the author constructs tables containing all numbers of the form $4n-1$, $6n-1$, and $12n-1$. *W. Feller*.

Denjoy, Arnaud. Les permutations spéciales de la suite normale des entiers positifs. C. R. Acad. Sci. Paris 217, 121–124 (1943). [MF 10636]

The author considers representations of the integer n in the form

$$n=\binom{x_1}{2}+\binom{x_2}{2}+\cdots+\binom{x_k}{2}$$

satisfying, for all $i<k$,

$$\binom{x_1}{2}+\cdots+\binom{x_i+1}{2}>n.$$

He proves several theorems which cannot be briefly explained. This investigation is a chapter of a book in the course of publication, whose aim will be the arithmetization of the transfinite numbers of the second class.

P. Erdös (Ann Arbor, Mich.).

Jonah, Harold F. S. Congruences connected with the solution of a certain Diophantine equation. Bull. Amer. Math. Soc. 51, 137–147 (1945). [MF 11830]

Transformations of some congruences connected with Fermat's last theorem are obtained from a functional equation. *H. S. Zuckerman* (Seattle, Wash.).

Ljunggren, Wilhelm. Sätze über unbestimmte Gleichungen. Skr. Norske Vid. Akad. Oslo. I. no. 9, 53 pp. (1942). [MF 11189]

It was proved by Mordell [Proc. London Math. Soc. (2) 21, 415–419 (1923)] that the Diophantine equation

$$Ax^4+Bx^2+C=Dy^2,$$

where the left-hand side has no squared factor in x, has only a finite number of solutions; however, his method does not give the solutions themselves or a bound for their number. The author obtains bounds for the number of integral solutions for some special equations of this type. If the fundamental units of certain biquadratic fields are known, then all the solutions can be determined. For instance, the following theorems are proved. Let a and D be square-free integers with $(a,6)=(D,3)=1$. The four equations $A^2x^4+2Ax^2+4=Dy^2$ with $D>1$ and $A=a$ or $A=3a$ have together at most four positive integral solutions unless $a=1$, $D=7$. In this case, $x^4+2x^2+4=7y^2$ has the solutions $x=y=1$; $x=9$, $y=31$; $x=2$, $y=2$; $x=6$, $y=14$; and $9x^4-6x^2+4=7y^2$ has the solution $x=1$, $y=1$. The three equations $a^2x^4\pm2ax^2+2=Dy^2$ and $a^2x^4+1=Dy^2$ have together at most one positive integral solution. Some other similar theorems are proved. [It may be remarked that the special case $A=D=1$ was solved completely by T. Nagell [Norsk Mat. Forenings Skrifter (I) no. 17 (1927)].]

With the same method, the author obtains results on equations of the form $x^3+b^3=Dy^2$. For $b=1$, this equation was studied by Nagell [Tôhoku Math. J. 24, 48–53 (1924)] and by C. E. Lind [Untersuchungen über die rationalen Punkte der ebenen kubischen Kurven vom Geschlecht eins, Uppsala, 1940]. Nagell proved that this equation has only the solution $x=-1$, $y=0$ if D is an integer, different from 1, which is only divisible by primes of the form $12n+5$ or by 3. For $b=-1$, it was proved by Nagell [Norsk Mat. Forenings Skrifter (I) no. 13 (1923)] that the equation has no solution for which x is odd if the class number h of the quadratic field $P(\sqrt{-D})$ is relatively prime to 3. However, Nagell was not able to solve the equation if $h\equiv0 \pmod 3$ or to determine the solutions with even x. For instance, he obtained certain results for $D=23$ and he found three positive solutions with even x for $D=7$, but he could not decide whether there are further solutions. In this paper the author proves that this equation has no solution for $D=23$ and exactly three positive solutions for $D=7$. Moreover, he proves the following theorem. Let $D>2$ be a square-free integer which is not divisible by 3 or primes of the form $6n+1$; then the eight equations $x^3+b^3=Dy^2$, $x^3+b^3=3Dy^2$, where $b=\pm1$ and $b=\pm2$, have together at most one solution in positive integers x, y. *A. Brauer* (Chapel Hill, N. C.).

Pillai, S. S. On $a^X-b^Y=b^y\pm a^x$. J. Indian Math. Soc. (N.S.) 8, 10–13 (1944). [MF 11271]

In two previous papers [J. Indian Math. Soc. (1) 19, 1–11 (1931); same J. (N.S.) 2, 119–122, 215 (1936)] the author proved that for given positive integers a and b the equation $a^x-b^y=c$ has only a finite number of solutions x, y for any c and at most one solution for sufficiently large c. It follows at once that $a^x-b^y=a^z-b^w$ has only a finite number of solutions x, y, z, w. In this paper the author intends to prove the same result for $a^x-b^y=b^z\pm a^w$. However, this result is correct only if $(a,b)=1$, although the

author also considers the case $(a, b)>1$. This can be seen by the following examples:

$$3^{k+1}-3^k=3^k+3^k \quad \text{and} \quad 2^{nk+2}-(2^n)^k=(2^n)^k+2^{nk+1},$$

where $a=b=3$ and $a=2$, $b=2^n$, respectively. *A. Brauer.*

Pillai, S. S. On m consecutive integers. IV. Bull. Calcutta Math. Soc. **36**, 99–101 (1944). [MF 11842]

[The first three notes appeared in Proc. Indian Acad. Sci., Sect. A. **11**, 6–12, 73–80 (1940); **13**, 530–533 (1941); these Rev. **1**, 199, 291; **3**, 66.] The theorem of the third note of this series is proved in a much simpler way.

H. S. Zuckerman (Seattle, Wash.).

Erdös, P. On the least primitive root of a prime p. Bull. Amer. Math. Soc. **51**, 131–132 (1945). [MF 11828]

The best result on the size of the least primitive root g of a prime p is that of Hua, who proved that $g<2^{m+1}p^{\frac{1}{2}}$, where m denotes the number of different prime factors of $p-1$ [Bull. Amer. Math. Soc. **48**, 726–730 (1942); these Rev. **4**, 130]. Using the method of Brun, the author proves that $g<p^{\frac{1}{2}}(\log p)^{17}$ for large p. It is pointed out that this is better than Hua's result if m is large, but worse if m is small.

H. W. Brinkmann (Swarthmore, Pa.).

Pocklington, H. C. Quadratic and higher reciprocity of modular polynomials. Proc. Cambridge Philos. Soc. **40**, 212–214 (1944). [MF 11867]

The λ-ic characters of two polynomials with respect to each other and a prime p are compared by relating each to the resultant of the polynomials. The ratio of the characters is shown to be $(-1)^{mn(p-1)/\lambda}$, where m and n are the degrees of the polynomials (which are assumed to be irreducible modulo p). Finally it is shown that the ordinary quadratic character of one prime with respect to another is the resultant of two polynomials. *I. Niven.*

Hua, Loo-Keng. On the distribution of quadratic non-residues and the Euclidean algorithm in real quadratic fields. I. Trans. Amer. Math. Soc. **56**, 537–546 (1944). [MF 11496]

Let p be a prime and q_1, q_2 and q_3 the three least prime quadratic nonresidues mod p. It was proved by Vinogradov [Trans. Amer. Math. Soc. **29**, 218–226 (1927)] that $q_1<p^{1/(2\sqrt{e})}(\log p)^2$ for sufficiently large p. Using Vinogradov's method and results of Rosser [Amer. J. Math. **63**, 211–232 (1941); these Rev. **2**, 150], the author obtains $q_1<(60p^{\frac{1}{2}})^{8/9}$, $q_2<(240p^{\frac{1}{2}})^{8/9}$, $q_3<(720p^{\frac{1}{2}})^{8/9}$ for $p>e^{250}$. By these inequalities he proves that there is no Euclidean algorithm in the quadratic field $\mathrm{P}(p^{\frac{1}{2}})$ if $p>e^{250}$. This improves the result of Erdös and Ko [J. London Math. Soc. **13**, 3–8 (1938)] that only a finite number of such fields with Euclidean algorithm exists. *A. Brauer.*

Hua, Loo-Keng, and Min, Szu-Hoa. On the distribution of quadratic non-residues and the Euclidean algorithm in real quadratic fields. II. Trans. Amer. Math. Soc. **56**, 547–569 (1944). [MF 11497]

[Cf. the preceding review.]

The problem of the existence of the Euclidean algorithm in quadratic fields $\mathrm{P}(m^{\frac{1}{2}})$ is still unsolved only in the case that $m=p$ is a prime. The algorithm does not exist if $p\equiv 3 \pmod 4$ and $p>19$ [E. Berg, Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] **5**, no. 5 (1935)], if $p\equiv 5 \pmod{24}$ and $p>29$ [Behrbohm und Rédei, J. Reine Angew. Math. **174**, 192–205 (1936)], and if $p\equiv 13 \pmod{24}$ and $p>109$ [A. Brauer, Amer. J. Math. **62**, 697–716 (1940); these Rev. **2**, 146]. In this paper the authors prove the same for $p\equiv 17 \pmod{24}$ with $p>137$. In a paper unknown to the authors, the same result was published by Rédei [Math. Ann. **118**, 588–608 (1942); these Rev. **6**, 38]. The proofs are entirely different. Rédei proved also that the algorithm does not exist for $p=89$, 113, 137, 61, and 109. *A. Brauer.*

Vandiver, H. S. New types of relations in finite field theory. Proc. Nat. Acad. Sci. U.S.A. **31**, 50–54 (1945). [MF 11781]

In two recent papers [Proc. Nat. Acad. Sci. U.S.A. **30**, 362–367, 368–370 (1944); these Rev. **6**, 117] the author has developed criteria for the number of roots of an equation in a finite field. In the present paper he applies these to equations where the number of roots is known, such as binomial equations, and thus derives relations in the field between binomial coefficients. A typical relation of this sort is the following. If $p^n-1=mc$, p a prime, $k>0$, and a is any nonzero element of a finite field $F(p^n)$, then, for $0<r<c$,

$$\sum_{s=0}^{\infty} c\binom{k(p^n-1)}{r+cs}a^{cs}=(-1)^{r+1}a^r \quad \text{or} \quad 0,$$

according as $(-a)^c=1$ or $(-a)^c\neq 1$. *H. W. Brinkmann.*

Vandiver, H. S. Fermat's quotient and related arithmetic functions. Proc. Nat. Acad. Sci. U.S.A. **31**, 55–60 (1945). [MF 11782]

If p is an odd prime and $n<p-1$, it is well known that

$$1^n2^n\cdots(p-1)^n\equiv pb_n \pmod{p^2},$$

where b_n is a Bernoulli number defined by the symbolic relationship $(b+1)^n=b^n$ $(b^k=b_k)$. The author proves the following generalization of this fact. If p is an odd prime with $p-1=mc$, let $M(c, n, p)$ be defined to be $p^{-1}\sum r^n$, where the summation extends over all integers in the set $1, 2, \cdots, p-1$ such that $r^c\equiv 1 \pmod p$, with $n\not\equiv 0 \pmod c$; then, if c and n are even,

$$M(c, n, p)\equiv -cn\sum_{k=0}^{m-1} b_{kc+n}/(kc+n) \pmod p.$$

For $c=p-1$, this relation becomes the congruence quoted at the beginning. *H. W. Brinkmann* (Swarthmore, Pa.).

Banerjee, D. P. On some formulae in analytical theory of numbers. II. Bull. Calcutta Math. Soc. **36**, 107–108 (1944). [MF 11844]

Continuation of a previous paper by the same author [Bull. Calcutta Math. Soc. **36**, 49–50 (1944); these Rev. **6**, 39]. A typical result is

$$\zeta^k(s)\prod_p(1-\tfrac{1}{2}(k-2)p^{-s})=\sum_1^{\infty} n^{-s}d(n)d(n^{1/k}).$$

L. Carlitz (Durham, N. C.).

Vinogradoff, I. M. General theorems on the estimations of trigonometric sums. C. R. (Doklady) Acad. Sci. URSS (N.S.) **43**, 47–48 (1944). [MF 11607]

New improvements in the author's well-known estimations of trigonometrical sums are announced. The main theorem is as follows. Let m, Q, P be integers, $m>0$, $P>0$,

$$F(x)=a_{n+1}x^{n+1}+\cdots+a_1x, \qquad n\geqq 11,$$

$a_{n+1}, \cdots, a_1$ real numbers, one of which, a_s, satisfies

$$a_s=aq^{-1}+\theta q^{-2}, \qquad (a, q)=1;\ |\theta|\leqq 1.$$

Then

$$\left|\sum_{z=Q+1}^{Q+P} e^{2\pi i m F(z)}\right| \leqq CP^{1-\rho_r} m^{2\rho_r},$$

where C is an absolute constant and $\rho_1=\tau/3n^2 \log(5n\tau^{-1})$ if $1<q\leqq P$, $q=P^\tau$, $\rho_2=1/3n^2\log(5n)$ if $P\leqq q\leqq P^s$, $\rho_3=\tau/3n^2\log(5n\tau^{-1})$if $P^s\leqq q\leqq P^{s+1}$, $q=P^{s+1-\tau}$. Proofs are to be published in a forthcoming book. *E. C. Titchmarsh.*

ANALYSIS

Dinghas, Alexander. Über eine algebraische Identität zwischen dem arithmetischen und geometrischen Mittel von n positiven Zahlen. Math. Z. 49, 563–564 (1944). [MF 11989]

The identity in question is

$$a_1+\cdots+a_n-nG_n=\sum_{k=2}^{n}\{a_k-kG_k+(k-1)G_{k-1}\},$$

where G_k is the geometric mean of $a_1, \cdots, a_k$. The right side is nonnegative, being of the form

$$x^k-kxy^{k-1}+(k-1)y^k \\ =(x-y)^2[x^{k-2}+2x^{k-3}y+\cdots+(k-1)y^{k-2}].$$

I. Kaplansky (New York, N. Y.).

Bödewadt, U. T. Zur Iteration reeller Funktionen. Math. Z. 49, 497–516 (1944). [MF 11992]

Let $f^{[1]}(x)=f(x)$, $f^{[n]}(x)=f(f^{[n-1]}(x))$, $n=2, 3, \cdots$. If $\varphi(x)$ is a solution of Abel's equation

$$(1)\qquad 1+\varphi(x)=\varphi(f(x)),$$

then

$$(2)\qquad f^{[k]}(x)=\varphi^{-1}(k-\varphi(x)),$$

and (2) may be taken as the definition of a kth iterate $f^{[k]}(x)$ even for fractional and negative k; $f^{[k]}(x)$ will in general depend on $\varphi(x)$ when k is not an integer. The author investigates the possibility of obtaining solutions of (1), and hence iterates of $f(x)$, having the same differentiability properties as $f(x)$. The functions $f(x)$ considered belong to the class S having the following properties: $f(x)$ is real, continuous and strictly increasing in $a\leqq x\leqq b$; $f(x)>x$; and $a\leqq f(a)<f(b)\leqq b$. A function $\varphi(x)$ will be called an A-function ("Abelsche Stufungsfunktion") for $f(x)$ if $\varphi(x)$ is real, continuous and strictly increasing in $f(a)\leqq x\leqq f(b)$ and satisfies (1). It is known that all A-functions $\psi(x)$ with $\psi(x_0)=0$ can be obtained from a given A-function $\varphi(x)$ with $\varphi(x_0)=0$ in the form $\psi(x)=s(\varphi(x))$, where $s(0)=0$, $s(t-1)=s(t)-1$, and $s(t)$ is real, continuous and strictly increasing. A trivial A-function is $\varphi_0(x)$, which is defined by $\varphi_0(x)=(x-x_0)/(x_1-x_0)$ in (x_0, x_1), $x_1=f(x_0)$, and by means of (1) in (x_1, x_2), (x_2, x_3), $\cdots$ and (x_{-1}, x_0), (x_{-2}, x_{-1}), $\cdots$, where $x_{i+1}=f(x_i)$. Starting from $\varphi_0(x)$, the author determines a sequence of A-functions $\varphi_n(x)$ of the form $\varphi_n(x)=s_n(\varphi_{n-1}(x))$, where $s_n(x)$ is defined by means of Bernoulli polynomials; if $f(x)$ is of class C^r (r continuous derivatives), $\varphi_r(x)$ is also of class C^r, and hence $f^{[k]}(x)$ is of class C^r for every integral or fractional k. If $f(x)$ is of class C^∞, $\varphi_n(x)$ converges to an A-function $\varphi(x)$ of class C^∞, and so all $f^{[k]}(x)$ are of class C^∞.

The author also discusses the iteration of completely monotonic functions. He calls $f(x)$ "completely increasing" if $(-1)^n D^n f(x)<0$ $(n=1, 2, \cdots; D=d/dx)$. If $f(x)\epsilon S$ and $\varphi(x)$ is an A-function such that

$$(3)\qquad \operatorname{Det}\binom{i}{k-1}D^{i-k+2}\varphi(x)>0$$

for $i, k=1, \cdots, n$ and $n=2, 3, \cdots$, then all iterates $f^{[k]}(x)$ with negative k are completely increasing. The author conjectures conversely that, if $f(x)\epsilon S$ and $f^{[-1]}(x)$ is completely increasing, then there is exactly one A-function $\varphi(x)$ satisfying (3); if this is true, the requirement of having completely increasing iterates of negative order makes the iterates uniquely determined. An equivalent conjecture is that if $g(x)$ is completely increasing there is exactly one completely increasing $h(x)$ such that $h(h(x))=g(x)$.

R. P. Boas, Jr. (Providence, R. I.).

Ghermanescu, Michel. Sur les valeurs moyennes des fonctions. Math. Ann. 119, 288–320 (1944). [MF 11901]

The author considers a function $u=u(P)$, where P is a point in p-dimensional space. Starting with the mean $\mu_0=\mu_0(u, P, R)$ of u over the spherical surface of radius R about the point P, the author defines higher successive means μ_i by the recursion formula

$$\mu_i(u, P, R)=(p+2i-2)R^{-(p+2i-2)}\int_0^R R^{p+2i-3}\mu_{i-1}(u, P, R)dR;$$

μ_1 is the mean on the homogeneous solid sphere. The μ_i satisfy the partial differential equation

$$\Delta\mu_k=\frac{\partial^2\mu_k}{\partial R^2}+\frac{p+2k-1}{R}\frac{\partial\mu_k}{\partial R},$$

where Δ is the Laplacian operator. The author proves a number of theorems characterizing the functions for which the means satisfy various linear relations, which may be considered as generalizations of Gauss's law of the spherical mean. The following is typical of the results obtained. Let $u(P)$ be a continuous function for which there exist functions $\phi_1(R), \cdots, \phi_m(R)$, such that

$$u(P)=\phi_1(R)\mu_1(u, P, R)+\cdots+\phi_m(R)\mu_m(u, P, R)$$

identically in P and R, every m of the μ_i being linearly independent. Then u satisfies a partial differential equation of the form

$$\Delta^m u+\lambda_1\Delta^{m-1}u+\cdots+\lambda_m u=0$$

with constant λ_i. In addition each μ_k can be expanded in the form

$$\mu_k(u, P, R)=\phi_1{}^k(r)u_1(P)+\cdots+\phi_m{}^k(r)u_m(P).$$

Here $r=R^2$, and the $\phi_i{}^k(r)$ are solutions (finite for $r=0$) of the ordinary differential equation

$$B_m{}^k(\phi)+\lambda_1 B_{m-1}^k(\phi)+\cdots+\lambda_m=0,$$

where $B_n{}^k$ is the differential operator defined by

$$B_n{}^k(v)=(d^n/dr^n)[r^{n+1-\frac{1}{2}p-k}((d^n/dr^n)r^{\frac{1}{2}p+k-1}v)].$$

F. John (Aberdeen, Md.).

Fourier Series and Generalizations, Integral Transforms

Neder, Ludwig. Ein Satz über die absolute Konvergenz der Fourier-Reihe. Math. Z. 49, 644–646 (1944). [MF 11983]

It is known that the well-known result of S. Bernstein concerning the absolute convergence of the Fourier series

of functions of the class Lip α, $\alpha>\frac{1}{2}$, remains valid if the modulus of continuity $\omega(\delta)$ of the function satisfies the condition $\int_0^1 t^{-\frac{3}{2}}\omega(t)dt<\infty$ [Hille and Tamarkin, Math. Ann. **108**, 525–577 (1933); S. Bernstein, C. R. Acad. Sci. Paris **199**, 397–400 (1934)]. The author proves the result in the case where

$$\omega(\delta)=O(\delta^{\frac{1}{2}}/\log\delta^{-1}\log_2\delta^{-1}\cdots(\log_k\delta^{-1})^{1+\epsilon}),\qquad \epsilon>0.$$

A. Zygmund (South Hadley, Mass.).

Arbault, Jean. Sur la convergence absolue des séries trigonométriques. C. R. Acad. Sci. Paris **217**, 592–594 (1943). [MF 11674]

The main results of the paper are as follows. (1) If the series $\sum\rho_n\cos(nx+\alpha_n)$ converges absolutely in a set E and if $x_0\epsilon E$, $x\epsilon E$, then the series $\sum\rho_n\sin n(x-x_0)$ converges absolutely. [For this result, see also Salem, Duke Math. J. **8**, 317–334 (1941); these Rev. **2**, 360.] (2) If $x\epsilon E$, $x'\epsilon E$, the set E is invariant under a symmetry with respect to $y=\frac{1}{2}(x+x')$. Some other properties of the set H of points y are indicated. *R. Salem.*

Wang, Fu Traing. Strong summability of Fourier series. Duke Math. J. **12**, 77–87 (1945). [MF 12071]

Let $f(x)$ be a function of period 2π and of the class L^r. Let $S_n(x)$ be the nth partial sum of the Fourier series of f, and let $2\varphi(t)=f(x+t)+f(x-t)-2f(x)$. For $r>1$, Hardy, Littlewood and Carleman proved that

$$(*)\qquad \sum_{\nu=1}^{n}|S_\nu(x)-f(x)|^2=o(n)$$

at every point x where

$$\int_0^h|\varphi(t)|^r dt=o(h),\qquad h\to 0.$$

[See, for example, the reviewer's Trigonometrical Series, Warsaw-Lwów, 1935, pp. 237–241.] The result is false for $r=1$ [Hardy and Littlewood, Fund. Math. **25**, 162–189 (1935)], though as recently shown by Marcinkiewicz [J. London Math. Soc. **14**, 162–168 (1939)] the relation (*) holds almost everywhere for every function f merely integrable. Hardy and Littlewood conjectured that if $|f|\log^+|f|$ is integrable the relation (*) is valid at every point x where

$$\int_0^h|\varphi(t)|\{1+\log^+|\varphi(t)|\}dt=o(h),\qquad h\to 0.$$

This conjecture is proved in the present paper.

A. Zygmund (South Hadley, Mass.).

Amerio, Luigi. Su una questione relativa all'analisi periodale. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **4**, 120–127 (1943) = Ist. Naz. Appl. Calcolo (2) no. **149**. [MF 11525]

The "periodic analysis" of a function $\phi(t)$ which is continuous in the interval $0\leqq t\leqq T$ is determined from a function of the type

$$\phi_n(t)=\sum_{k=0}^{r_n}P_{k,n}(t)e^{i\lambda_{k,n}},$$

where the $\lambda_{k,n}$ are real or complex constants and the $P_{k,n}(t)$ are polynomials such that, in the interval $0\leqq t\leqq T$,

$$\phi(t)=\lim_{n\to\infty}\phi_n(t).$$

Suppose now that $\phi_n(t)$ is an integral of a linear homogeneous differential equation with constant coefficients whose order does not exceed R. The author proves that $\phi(t)$ is a function of the same class. *A. E. Heins.*

Lorentz, Georg Gunther. Über die Mittelwerte der Funktionen eines Orthogonalsystems. Math. Z. **49**, 724–733 (1944). [MF 11980]

Let $\{\phi_n(t)\}$ be an orthonormal system and $\{a_n\}$ a bounded sequence of real numbers. Denote by σ_n^α the Cesàro means of the sequence $\{a_n\phi_n(t)\}$. The author shows: (1) $\sigma_n^\alpha\to 0$ if $\alpha>\frac{1}{2}$; (2) for $\alpha=\frac{1}{2}$ the statement does not hold; (3) for the special Rademacher system

$$\phi_n(t)=\operatorname{sign}\sin(2^{n+1}\pi t)$$

the statement remains true for all $\alpha>0$. *W. Feller.*

Agnew, Ralph Palmer. Criteria for completeness of orthonormal sets and summability of Fourier series. Duke Math. J. **11**, 801–821 (1944). [MF 11582]

In the present paper the author states in clear form the fundamental facts about completeness of orthogonal sets. He proves a considerable number of theorems of interest and direct bearing on completeness; these are without significance unless precisely stated and space does not permit that in this review. Agnew then defines a kernel $K(x,y,t)$ associated with a set of convergence factors. The main problem of the paper is the connection of these kernels with completeness and the development of criteria involving them. There are a number of theorems, in particular, theorems involving convergence in the mean and summability. There is a brief treatment of Fourier series.

T. Fort (Bethlehem, Pa.).

Wintner, Aurel. Gibbs' phenomenon and the prime number theorem. Amer. J. Math. **67**, 167–172 (1945). [MF 11932]

If p denotes a prime and $S(x)=\int_0^x u^{-1}\sin u\,du$, then

$$\sum_{p\leqq m}p^{-1}\sin(t\log p)-S(t\log m)$$

tends to a finite limit as $m\to\infty$, uniformly in every finite interval $|t|<T$. It should be noted that $S(x)$ is bounded.

S. Bochner (Princeton, N. J.).

Wintner, Aurel. Diophantine approximations and Hilbert's space. Amer. J. Math. **66**, 564–578 (1944). [MF 11395]

This paper deals with the completeness in Hilbert space of certain sequences of functions of importance in number theory and with transformations in Hilbert space based on the sieve of Eratosthenes. The sequence $\{1,\varphi(nt)\}$ is shown to be L^2-complete (closed) on the interval $0\leqq t\leqq\frac{1}{2}$ if

$$\phi(t)\sim\sum_{k=1}^{\infty}k^{-\lambda}\cos 2\pi kt,\qquad \Re(\lambda)>\tfrac{1}{2};$$

in particular, this is true for $\phi(t)=t-[t]$. For more general functions whose Fourier coefficients are $O(k^{-\frac{1}{2}-\epsilon})$, it is shown that $\sum c_n\phi(nt)$ converges in the L^2 sense whenever $c_k=O(k^{-\frac{1}{2}-\epsilon})$.

It is shown from the theory of bounded infinite matrices that the transformation

$$y_k=\sum_{d|k}\lambda_{k/d}x_d$$

takes each point $(x_1,x_2,\cdots)$ in Hilbert space into a point $(y_1,y_2,\cdots)$ of Hilbert space if and only if $\sum\lambda_k k^{-s}$ is

regular and bounded in $\Re(s)>0$; this implies that $c_k=O(k^{-\frac{1}{2}-\epsilon})$ in the above theorem cannot be weakened to $\sum|c_k|^2<\infty$ unless $\sum a_k k^{-s}$ and $\sum b_k k^{-s}$ are regular and bounded in $\Re(s)>0$.

Finally, let $\sum_p$ denote summation over all primes and $(k;l)=kl(\text{g.c.d. of } k \text{ and } l)^{-2}$. Then it is stated (among other related theorems) that, if $\lambda_1, \lambda_2, \cdots$ is a real nontrivial sequence satisfying $\lambda_{mn}=\lambda_m\lambda_n$ and $\sum_p\lambda_p^2<\infty$, then $\{\lambda_{(k;l)}\}$ is a bounded matrix if and only if $\prod_p[1+(p^s-1)^{-1}\lambda_p]$ is regular and bounded in $\Re(s)>0$. [It appears to the reviewer that the product should be $\prod_p[1+(p^s-\lambda_p)^{-1}\lambda_p]$. We note also that $b=n^{-\frac{1}{2}}\log(n+1)$ on line 3 of page 570 should be $b_n=n^{-\frac{1}{2}}/\log(n+1)$.] *R. H. Cameron.*

Wintner, Aurel. The moment problem of enumerating distributions. Duke Math. J. **12**, 23–25 (1945). [MF 12066]

Let

$$(*)\qquad \mu_n=\int_0^\infty x^n d\alpha(x), \qquad n=0, 1, 2, \cdots,$$

for at least one nondecreasing $\alpha(x)$ with $\alpha(0)=0$, $\alpha(\infty)=1$. If the series

$$s=\sum_{n=1}^\infty(-4\pi^2)^n\mu_{2n}/(2n)!$$

converges absolutely, then according as $s\neq 0$ or $s=0$ there exists no solution or just one solution $\alpha(x)$ of (*) such that the only points of increase of $\alpha(x)$ are at the nonnegative integers. *R. P. Boas, Jr.* (Providence, R. I.).

Bellman, Richard. Some applications of the Fourier integral to generalized trigonometric series. Duke Math. J. **11**, 703–713 (1944). [MF 11573]

In establishing a "high indices" theorem for series of the form

$$f(s)=\sum a_j e^{-l_j s}, \qquad \sigma>0,$$

the chief difficulty generally comes in establishing first that under suitable conditions $a_j=o(1)$ or $O(1)$ as $j\to\infty$. The author obtains these results under the familiar assumption that $l_n-l_{n-1}\geqq 2c>0$ and the rather unusual one that

$$\int_{-\infty}^{\infty}|t^{-1}\sin\sigma t|^p|f(\sigma+it)|^p dt\leqq K^p, \qquad 1\leqq p\leqq 2,$$

where K is independent of σ near zero. The apparatus set up is then applied to obtain extensions of theorems on generalized trigonometric series. *H. Pollard.*

Hornich, Hans. Über gewisse trigonometrische Integrale. II. Math. Z. **49**, 374–379 (1944). [MF 11998]

[Part I appeared in Math. Z. **48**, 785–791 (1943); these Rev. **5**, 3.]

Some remarks, hardly worth quoting, about positive definite functions which are given as convergent infinite products

$$\prod_{i=1}^\infty(\sin a_i x)/(a_i x).$$

S. Bochner (Princeton, N. J.).

Amerio, Luigi. Su alcune questioni relative alla trasformazione di Laplace. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) **7(76)**, 26 pp. (1943) = Ist. Naz. Appl. Calcolo (2) no. **154**. [MF 11729]

The author discusses some of the criteria which enable one to recognize that a function $f(p)$ is the Laplace transform of a function $F(t)$ which possesses a certain number of derivatives. A generalization of the Heaviside expansion theorem is also discussed. *A. E. Heins.*

Bayard, Marcel. Sur une utile extension de la transformation de Laplace. C. R. Acad. Sci. Paris **217**, 471–472 (1943). [MF 11666]

A remark, which seems to be very familiar, comparing the inversion of unilateral and bilateral Laplace integrals. *R. Salem* (Cambridge, Mass.).

Hecke, E. Herleitung des Euler-Produktes der Zetafunktion und einiger *L*-Reihen aus ihrer Funktionalgleichung. Math. Ann. **119**, 266–287 (1944). [MF 11900]

The main purpose of this paper is to obtain the expansions of $\eta(\tau)$, $\vartheta(\tau)$, $\eta^2(\tau)$ and $\eta^4(\tau)/\vartheta(\tau)$ from their functional properties alone. These functions are modular forms of dimension half an odd integer. The expansions are of the form $\sum_{n=0}^\infty a(n)z^n$, with $z=e^{2\pi i\tau/\lambda}$, λ a suitable divisor of 24. Corresponding to this expansion, the Euler product expansion of the function $\varphi(s)=\sum_{n=1}^\infty a(n)n^{-s}$ is also determined. The author has previously shown [Math. Ann. **112**, 664–699 (1936)] that if $\varphi(s)$ has the functional properties of $\zeta(2s)$ then the corresponding modular form has the functional properties of $\vartheta(\tau)$. Therefore the Euler expansion of $\zeta(s)$ is obtained from the functional properties of $\zeta(s)$. The functions $\varphi(s)$ corresponding to the remaining modular forms considered are various *L*-series. *H. S. Zuckerman* (Seattle, Wash.).

Differential Equations

Fadle, Johann. Über Kurven konstanten Bahndruckes. Z. Angew. Math. Mech. **21**, 118–123 (1941). [MF 12147]

The problem reduces to that of integrating the differential equation

$$2(H-y)(1+y'^2)^{-1}y''=-1+C(1+y'^2)^{\frac{1}{2}},$$

where H and C are constants. The solutions are studied in detail. *W. Feller* (Ithaca, N. Y.).

Levinson, Norman. Transformation theory of non-linear differential equations of the second order. Ann. of Math. (2) **45**, 723–737 (1944). [MF 11377]

The author considers the differential equation

$$\dot{y}+f(x, y)y+g(x)=e(t), \qquad y=\dot{x},$$

where $e(t)$ has period L, and points out how the equation leads to a transformation T of the xy-plane onto itself and to curve-families on the torus, to which work of Poincaré, G. D. Birkhoff, Denjoy and others can be applied. It is shown that, if every solution eventually remains within a fixed circle as $t\to\infty$, then there exists at least one periodic solution with period L. The transformation T of the xy-plane is studied in the neighborhood of fixed points by means of power series; the fixed points are classified according to stability and equations relating the numbers of fixed points of different types are established. *W. Kaplan* (Providence, R. I.).

Erdélyi, A. Certain expansions of solutions of the Heun equation. Quart. J. Math., Oxford Ser. **15**, 62–69 (1944). [MF 11802]

The subject is the expansion of a solution of the Heun equation

$$z(z-1)(z-a)\left\{\frac{d^2y}{dz^2}+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}+\frac{\epsilon}{z-a}\right)\frac{dy}{dz}\right\}+\alpha\beta(z-h)y=0,$$

with $\alpha+\beta-\gamma-\delta-\epsilon+1=0$, in series of the form $\sum_{m=0}^{\infty} c_m P_m$, the terms P_m being solutions of the hypergeometric equation

$$z(z-1)\left\{\frac{d^2P_m}{dz^2}+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}\right)\frac{dP_m}{dz}\right\}+(\lambda+m)(\mu-m)P_m=0,$$

with the same γ and δ, and with $\lambda+\mu-\gamma-\delta+1=0$. The expansion is possible for certain values of λ and μ, with appropriate determination of the functions P_m. There are two types of expansion. Their regions of convergence and their interrelations are discussed. For special values of the accessory parameter h the equation admits a solution which is regular at two of the equation's singular points. This is called a Heun function. The expansion of Heun functions is especially considered. The paper is related to an earlier one by the same author [Duke Math. J. 9, 48–58 (1942); these Rev. 3, 241]. *R. E. Langer.*

Batchelor, G. K. Power series expansions of the velocity potential in compressible flow. Quart. Appl. Math. 2, 318–328 (1945). [MF 11773]

The flow of a compressible fluid around a fixed obstacle leads to a nonlinear second order boundary value problem in two variables r and θ for the velocity potential φ. Treating first the case of flow without circulation, the author tries to find the solution in the form

$$\varphi=r\cos\theta+r^{-1}f_1(\theta)+r^{-2}f_2(\theta)+\cdots. \tag{1}$$

The validity of this expansion, if it converges, depends on the periodicity of the f_n. In this direction the periodicity of f_1 and f_2 is proved, and (up to the order M^2) that of all f_n (M is a parameter appearing in the differential equation, the so-called Mach number). Moreover the author asserts that in carrying out the iteration process in powers of M he has been able to show that aperiodicity can not arise for f_3 and f_4 (and in case of a doubly symmetric body also not for f_5) if terms of the order M^4 are considered, nor when terms of the order M^6 in f_3 are considered. Numerical computations of I. Imai [Proc. Phys.-Math. Soc. Japan 23, 180–193 (1941)] show that in the case of a circular cylinder the solution (1) holds at least up to the order M^6. In case of nonzero circulation the author starts again with the solution (1) augmented by an aperiodic term $f_0(\theta)$. Here, however, other terms turn out to be aperiodic and the solution is amended to the form

$$\varphi=r\cos\theta+f_0(\theta)+\sum_{n=1}^{\infty} r^{-n}f_n(\theta)+Mk_0^2\log r\,\partial\varphi_0/\partial y,$$

correct to the order M^2 at least, where φ_0 is the velocity for incompressible flow; f_0 and f_1 are discussed in more detail. Connections with the Hill and Mathieu equations are established. *E. H. Rothe* (Ann Arbor, Mich.).

Strutt, M. J. O. Reelle Eigenwerte verallgemeinerter Hillscher Eigenwertaufgaben 2. Ordnung. Math. Z. 49, 593–643 (1944). [MF 11984]

The separation of the scalar Helmholtz equation in paraboloidal coordinates leads to differential equations of the form

$$w''+[\lambda+\gamma_1\Phi_1(z)+\gamma_2\Phi_2(z)]w=0, \tag{1}$$

while separation in ellipsoidal coordinates leads to

$$w''+[\lambda+\gamma_1\Phi_1(z)+\gamma_2\Phi_2(z)+\gamma_3\Phi_3(z)]w=0. \tag{2}$$

In both of these equations λ and γ_n are real parameters and Φ_n are real periodic functions of period ζ. The author first studies the relationships among λ, γ_1, γ_2 in equation (1) under the boundary conditions $w(z_0+\zeta)=\sigma w(z_0)$ and $w'(z_0+\zeta)=\sigma w'(z_0)$. Here σ is real, or complex with $|\sigma|=1$. When $\sigma=1$, the solution is periodic; if $\sigma=-1$, the solution is semi-periodic. To obtain the adjoint boundary conditions replace σ by $1/\sigma$. Lower bounds for $(\gamma_1^2+\gamma_2^2)^{-1}$ in terms of λ, ζ, σ can easily be established using the Green's function for the operator $d^2/dz^2+\lambda$ satisfying the boundary conditions given for w. By letting $\gamma_n=\mu_n\gamma$, the Sturm-Liouville theory can be applied to obtain other bounds on γ.

If λ, γ_1, and γ_2 are considered to form the three axes of a rectangular coordinate system, the relation among λ, γ_1, γ_2 can be represented by a surface for each value of σ. The properties of these surfaces are investigated. The first and second derivatives of λ with respect to γ_1 and γ_2 are evaluated. It is shown that λ is a single-valued function of γ_1 and γ_2. Lines on these surfaces are distinguished according as $\psi=1+\Phi_1\tan\varphi_1+\Phi_2\tan\varphi_2$ is greater or less than zero, where $\tan\varphi_n=(\gamma_n-\Gamma_n)/(\lambda-\Lambda)$ when (Λ,Γ_n) are the coordinates of a point on the surface. If $\psi<0$, then for every real $\sigma\neq1$ there is a denumerably infinite number of both positive and negative λ's. If $\psi>0$, then for real $\sigma\neq1$ there is only a finite number of λ's. For both cases there is a denumerably infinite number of λ's when $|\sigma|=1$. The arrangement of the eigenvalues λ is investigated for $\psi>0$. It is shown that if, between two neighboring eigenvalues λ, both belonging to $\sigma=1$, or to $\sigma=-1$, there is an eigenvalue belonging to a real $\sigma\neq1$, there must be a second in the same range. Between two unlike periodic λ's, one belonging to $\sigma=1$, the other to $\sigma=-1$, there is one and only one λ belonging to a complex σ, $|\sigma|=1$.

The asymptotic behavior of the surfaces is considered for the case $|\lambda|+|\gamma_1|+|\gamma_2|\gg\zeta^{-2}$ by using the approximate "WKB" method as developed by R. E. Langer for the case where $\lambda+\gamma_1\Phi_1+\gamma_2\Phi_2$ has two zeros in the interval $(z_0, z_0+\zeta)$. Explicit formulas connecting σ, γ_1, γ_2, λ are derived. It is shown that, if $|\sigma|$ is nearly 1, then

$$\gamma_1\partial\lambda/\partial x_1+\gamma_2\partial\lambda/\partial x_2<0,\quad \partial^2\lambda/\partial\gamma^2>0;$$

as $\lambda\to-\infty$, $\partial^2\lambda/\partial\gamma^2\to0$. The discussion is generalized to the case where $\lambda+\gamma_1\Phi_1+\gamma_2\Phi_2$ has more than two zeros.

Finally, the above discussions are repeated for cases where more than two Φ's are involved, which include equation (2) as a special case. *H. Feshbach* (Cambridge, Mass.).

Vessiot, Ernest. Sur les équations aux dérivées partielles de premier ordre considérées comme des équations de contact et, en particulier, sur l'intégration des équations semi-linéaires. Ann. Sci. École Norm. Sup. (3) 59, 211–273 (1942). [MF 11884]

Let $x_1, \cdots, x_{n+1}$ be the coordinates of a space E, and $p_1, \cdots, p_{n+1}$ the homogeneous coordinates of a space G. An equation

$$\Phi(x_1, \cdots, x_{n+1}; p_1, \cdots, p_{n+1})=0, \tag{1}$$

homogeneous in the coordinates $p_1, \cdots, p_{n+1}$, is called an equation of contact. It is an algebraic problem to determine the functions

$$z_{ij}(x_1, \cdots, x_{n+1}; y_1, \cdots, y_a)$$

so that the s equations

$$X_h\equiv p_h+z_{ih}p_i=0,\ h=1, \cdots, s;\ i=1, \cdots, r;\ s+r=n+1, \tag{2}$$

satisfy (1) ($y_1, \cdots, y_a$, if present, standing for arbitrary functions of some of the x's). On the other hand, an s-dimensional variety

$$x_{s+i}=f_i(x_1, \cdots, x_s) \tag{3}$$

produces equations of type (2) when the $dx_{s+i}=(\partial f_i/\partial x_k)dx_k$ $(k=1, \cdots, s)$ are substituted in the equation of Pfaff

$$(4) \qquad p_1dx_1+\cdots+p_{n+1}dx_{n+1}=0.$$

If there exists a variety which, through equation (4), produces equations (2), it is said to be an integral of the equation of contact (1). The first part of the paper is devoted to the problem of existence of integrals of (1), especially those for which the dimension s of the variety is less than n. Treating the p_i as $\partial f/\partial x_i$, this problem reduces to finding complete subpencils (faisceaux) of degree s of the pencil of infinitesimal transformations

$$\{X_1, \cdots, X_s, \partial f/\partial y_1, \cdots, \partial f/\partial y_n\}.$$

In the second part, a complete integral of $\Phi=0$:

$$(5) \qquad G_i(x_1, \cdots, x_{n+1}; z_1, \cdots, z_{s-1})=z_{s+i-1}$$

is assumed to be known (the $z_1, \cdots, z_n$ are arbitrary constants). Properties of such integrals are given. In particular, looking on the z_i as variables, a transformation based on (5) is given which reduces $\Phi=0$ to the form $p_{n+1}=0$.

An equation of contact $\Phi=0$ is said to be semi-linear if it has a complete integral of s dimensions with $1<s<n$. In the final section of the paper the author treats the problem of recognizing whether or not an equation of contact is semi-linear, and, if it is, finding all of its complete integrals of s dimensions. *F. G. Dressel.*

Faedo, Sandro. L'unicità delle successive approssimazioni nel metodo variazionale. Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. (7) **13**, 679–706 (1942) = Ist. Naz. Appl. Calcolo (2) no. **126**. [MF 11518]

This paper deals with the uniqueness of the successive approximations used in the variational method of Picone for the solution of a system of linear partial differential equations with variable coefficients, for which the time interval involved is infinite. The author establishes sufficiency theorems for the uniqueness of extremals for a class of calculus of variations problems involving higher derivatives of the dependent variables when one of the limits of integration is infinite. *W. T. Reid* (Evanston, Ill.).

Faedo, Sandro. Sul metodo variazionale per l'analisi dei problemi di propagazione. Pont. Acad. Sci. Comment. **6**, 657–685 (1942) = Ist. Naz. Appl. Calcolo (2) no. **137**. [MF 11507]

This paper is concerned primarily with the application of the variational method of Picone to the solution of a system of linear partial differential equations. In particular, the author discusses the problem of determining conditions which insure that the calculus of variations problem which is involved is positive regular, thus extending some of his previous results [Ann. Scuola Norm. Super. Pisa (2) **10**, 139–152 (1941); these Rev. **3**, 245]. He also presents a modification of the variational problem which permits greater freedom in the choice of the complete systems of orthonormal functions involved. *W. T. Reid* (Evanston, Ill.).

Cimino, Massimo. Una soluzione in grande del problema di Cauchy per una particolare equazione in tre variabili, ottenuta con un metodo di M. Picone. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **2**, 800–809 (1941) = Ist. Naz. Appl. Calcolo (2) no. **103**. [MF 11511]

The problem of Cauchy for Laplace's equation $\Delta u=0$ with the conditions $u=f_0(x, y, z)$, $\partial u/\partial r=f_1(x, y, z)$ on the sphere $r=(x^2+y^2+z^2)^{\frac{1}{2}}=a$ can be solved by a series of the form

$$u=\sum_{m=0}^{\infty}\{r^mF_m(\theta, \phi)+r^{-m-1}G_m(\theta, \phi)\},$$

where F_m, G_m are spherical harmonics which are easily found from the expansions of f_0 and f_1 as series of spherical harmonics. When f_0 and f_1 satisfy suitable conditions, the solution is unique, analytic and regular in the whole of space save possibly the origin.

The present paper is concerned with the same problem for the equation $\Delta^n u=0$ with the conditions

$$\partial^k u/\partial r^k=f_k(x, y, z), \quad k=0, 1, \cdots, 2n-1,$$

on the sphere $r=a$. The method employed is a straightforward generalization of that used in the case $n=1$. The solution is again unique, analytic and regular in the whole of space save possibly the origin. *E. T. Copson* (Dundee).

The paper is identical with that reviewed in these Rev. 3, 128.

★Sternberg, Wolfgang J., and Smith, Turner L. The Theory of Potential and Spherical Harmonics. Mathematical Expositions, no. 3. University of Toronto Press, Toronto, Ont., 1944. xii+312 pp. $3.35.

Most advanced books on mathematics that have been published in America have been detailed treatments of specialized topics. There has been a lack of sound introductions to advanced topics not going into such detail that the reader is unable to see the woods for the trees. This gap, however, is being filled in part by the series "Mathematical Expositions" published under the auspices of the University of Toronto. The purpose of these books is to present the subject matter in such a way that the newcomer to the field can master the fundamental principles without having to consider all possible elaborations at the same time. The present book could very profitably be used as a text in an early graduate course.

Most of the material is presented in the terminology of the Newtonian potential, with suitable modifications to cover the case of logarithmic potential. Many of these modifications are given as exercises. After a rapid review of vectors, the first three chapters give definitions and elementary facts about the potential function. Topics discussed include the computation of potentials of volume, surface, and line distributions and double layers; force fields, harmonic functions and Laplace's equation; behavior of the potential function at infinity; and the integral theorems of Gauss, Green, and Stokes. Illustrations are given of the application of the theory to gravitational fields and to flow problems. In chapter IV the theory of Legendre polynomials and spherical harmonics is developed. A considerable amount of detail is given about these functions: orthogonality properties, recurrence formulas, addition theorems, theory of surface spherical harmonics.

Chapter V is a study of the behavior of the potential at points of the mass including Poisson's equation and continuity properties of the function and its derivatives at points of volume and surface distributions and double layers. Chapter VI discusses the relationship between the logarithmic potential and function theory. Chapter VII is a statement of the various types of boundary value problems. It also contains uniqueness theorems and expository material on the Dirichlet principle and the historical background of the problems. In chapter VIII, the Dirichlet problem for the circle is solved by means of the Poisson integral. A considerable amount of theory on Fourier series is developed. Green's function is discussed and applica-

tions to conformal mapping are made. Chapter IX contains Poisson's integral for the sphere and the expansion of functions in surface spherical harmonics. Chapter X gives the Fredholm theory of integral equations of the second kind. Although the presentation is elegant, the material would probably be more understandable to an inexperienced reader if there were some discussion of the motivation behind the method of solution, such as is given in the corresponding chapter of Kellogg's book. Chapter XI states the boundary value problems in such a way that the results of the preceding chapter apply to them and give their solution.

The book as a whole is clearly written and comfortably readable. Chapters IV and X are perhaps a bit crowded, but by no means obscure. It should fill very satisfactorily the purpose for which it was designed. *J. W. Green.*

Vasilesco, Florin. Sur quelques critères généraux de régularité et de stabilité. Ann. Sci. École Norm. Sup. (3) **59**, 275–295 (1942). [MF 11885]

In the criteria for the regularity of a point, given by de la Vallée Poussin [Acad. Roy. Belgique. Bull. Cl. Sci. (5) **24**, 368–384, 672–689 (1938)], the author substitutes arbitrary mass distributions for capacity distributions. He obtains the following result. If μ is a positive mass distribution on E whose potential v is bounded in the neighborhood of the point $P\epsilon E$ and if v_ρ is the potential of μ_ρ, the distribution on E_ρ (the part of E inside the sphere around P of radius ρ), which we obtain by sweeping μ onto E_ρ, then a necessary and sufficient condition that P be a regular point of E is that $v_\rho(P)=v(P)$ for all ρ. If P is not regular, then $\lim_{\rho\to 0} v_\rho(P)=0$. Similarly all other criteria of regularity can be generalized. Another way of generalizing the criteria is exemplified by the following. Let $\lambda_n>\lambda_{n+1}\to 0$ and let E_n be the part of E between the circles of radii λ_n and λ_{n+1} with centers at P. Furthermore, let μ_n be a positive mass distribution on E_n, causing a potential v_n, such that all v_n, $n=1, 2, \cdots$, have a common positive upper and lower bound. Then P is a regular point of E or not, according as $\sum \mu_n/\lambda^n$ diverges or converges. *František Wolf.*

Gelfand, I. S. Direct and inverse problem of the magnetic potential for a spheric segment. Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **8**, 292–294 (1944). (Russian. English summary) [MF 12140]

In the present article there are established formulae for the gravitation gradient and the differences of curvature of the underground strata having the form of a pressed or drawn out rotation ellipsoid. Further there are given formulae for the determination of all elements of the stratum of the rotation ellipsoid, according to the observed values of the gravitation gradient and the differences of curvature at a known abundant density of the rock.

Author's summary.

Gelfand, I. S. The inverse problem of the gravitational potential for the homogeneous ellipsoid generated by rotation. Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **8**, 295–297 (1944). (Russian. English summary) [MF 12141]

There are given formulae for the magnetic potential and the components of the magnetic force of the underground strata having the form of a spheric segment in the case of an homogeneous and direct magnetization.

A method, based on the solution of the direct problem, is suggested for the determination of the elements of the stratum of the spheric segment upon the observed values of horizontal and vertical components of the magnetic force at a known intensity of magnetization of rocks.

Author's summary.

Fichera, Gaetano. Sviluppi in serie e teoremi di decomposizione in somma per le funzioni iperarmoniche. Rend. Circ. Mat. Palermo **63**, 24 pp. (1941) = Ist. Naz. Appl. Calcolo (2) no. **109**. [MF 11510]

The paper continues earlier investigations by Almansi, Picone and Nicolesco. The author considers a hyperharmonic function u of order k, that is, a solution of the iterated Laplace equation (1) $\Delta^k u=0$ in $n+2$ independent variables. Using polar coordinates r and P, where P is restricted to the unit sphere, it is proved that every solution u of (1) which is of class C^{2k} in the spherical shell $r'<r<r''$ can be represented by a generalized "Laurent series" in that shell. With $\mu=k-(n+2)/2$, this Laurent series has the form

$$u(r, P)=\sum_{s=0}^{\infty}\sum_{i=1}^{k}\{c_{2i-1}^{(s)}(P)r^s+c_{2i}^{(s)}(P)r^{-s-n}\}r^{2i-2}$$

in case n is odd or in case n is even and $\mu<0$. If n is even and $\mu\geqq 0$, the series takes a more complicated form. The curious distinction in the two cases is due to the presence of multiple roots of the characteristic equation of the Euler differential operator

$$\{d^2/dr^2+r^{-1}(n+1)d/dr-s(s+n)/r^2\}^k$$

in the case of even n, when $0\leqq s\leqq \mu$. The Laurent series is used to derive a decomposition of the hyperharmonic function into simple harmonic functions. This decomposition for odd n and for even n with $\mu<0$ is of the "type of Almansi"

$$u(r, P)=\sum_{i=0}^{k-1} u_i(r, P)r^{2i},$$

where the u_i are uniquely determined harmonic functions. For even n with $\mu\geqq 0$, the decomposition is complicated by an additional sum of the form $\sum_{s=0}^{\mu}\Pi_s r^{2(\mu-s)}\log r$ (decomposition of the "type of Picone"), where the Π_s are harmonic polynomials of degree s, the harmonic functions u_i being no longer uniquely determined. *F. John.*

Fichera, Gaetano. Un teorema generale sulla struttura delle funzioni iperarmoniche. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **3**, 511–523 (1942) = Ist. Naz. Appl. Calcolo (2) no. **130**. [MF 11519]

[Cf. the preceding review.] This paper derives for more general regions A a decomposition of the "type of Picone" for the case of biharmonic functions in two independent variables ($k=2$, $n=0$). The connection between the occurrence of logarithmic terms in the decomposition and the topological structure of A is brought out clearly. The regions A admitted may be described roughly as formed by cutting m simply connected holes $B_1, \cdots, B_m$ out of a simply connected set B_0 in the xy-plane. Let $O=(a, b)$ denote an arbitrary point, and $O_1, O_2, \cdots, O_m$ arbitrary points in $B_1, \cdots, B_m$, respectively. Then for a function $u(P)$, which is biharmonic in A, there is a decomposition of the form

$$u(P)=u_0(P)+\overline{OP}^2u_1(P)+\sum_{i=1}^{m}[(\alpha_i+a\gamma_i)x+(\beta_i+b\gamma_i)y]\log\overline{O_iP},$$

where u_0 and u_1 are harmonic in A, and where the α_i, β_i, γ_i

are constants given by

$$\alpha_i=\int_{C_i}(\Delta u dx/dn-x d\Delta u/dn)ds,\quad \gamma_i=\int_{C_i}(d\Delta u/dn)ds,$$

$$\beta_i=\int_{C_i}(\Delta u dy/dn-y d\Delta u/dn)ds.$$

Here C_i denotes any regular simple curve in A, including B_i but none of the other B_j. *F. John* (Aberdeen, Md.).

Raymond, François. Applications de la transformée de Fourier à la résolution des problèmes de champ en électrotechnique. C. R. Acad. Sci. Paris 217, 499–501 (1943). [MF 11668]

With the aid of the Fourier transform theorem, the author solves the partial differential equation

$$\varphi_{xx}+\varphi_{yy}=-4\pi a(x)b(y),$$

where $a(x)$ and $b(y)$ are given functions inside a given rectangle and zero outside of this rectangle.

A. E. Heins (Cambridge, Mass.).

Vecoua, Elias. On integral representations of the solutions of differential equations. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 4, 843–852 (1943). (Georgian and Russian. English summary) [MF 11709]

Let D_p be a region in Euclidean space of $p+1$ dimensions, such that D_p is symmetric with respect to the hyperplane $y=0$, and such that any segment perpendicular to $y=0$ and joining two points of D_p lies entirely in D_p. For regions D_p a simple connection is established, in terms of integral equations, between the solutions of equations of the forms

$$L_0u\equiv Lu+\partial^2u/\partial y^2=0 \tag{1}$$

and

$$L_\lambda u\equiv L_0u+\lambda^2u=0, \tag{2}$$

where λ is a constant parameter and L is an arbitrary linear differential or functional operator with respect to $x_1, \cdots, x_p$, with coefficients independent of y. A derived connection holds between solutions of the nonhomogeneous equations $L_0u=f$ and $L_\lambda u=g$, where f and g are given continuous functions in D_p.

Let D_p be an infinite cylindrical region, let R be a linear operator, and let F be a given continuous boundary function for D_p. Consider the boundary problem of determining a regular solution in D_p of (1) or (2) satisfying $Ru=F$ on the boundary; for $Ru\equiv u$ and for $R\equiv d/dn$, we have the Dirichlet and Neumann problems, respectively. The above result is used to show that solutions of the boundary problems for (1) and (2) can be obtained from each other by quadratures. *E. F. Beckenbach* (Austin, Tex.).

Vekua, Ilja. On a new representation of solutions of differential equations. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 4, 941–950 (1943). (Georgian. Russian summary) [MF 11711]

Developments analogous to those of the paper reviewed above are made for regions V_p contained in the half-space $y>0$ and such that any segment perpendicular to $y=0$ from a point of the domain lies, except for the foot of the perpendicular, entirely in V_p. In the boundary value problem, the values of u and $\partial u/\partial y$ are given continuous functions on $y=0$, while the equation $Ru=F$ is to be satisfied on the rest of the boundary. *E. F. Beckenbach* (Austin, Tex.).

Chvedelidze, B. V. Poincaré's problem for a linear differential equation of second order of elliptic type. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] 12, 47–77 (1943). (Georgian. Russian summary) [MF 11685]

The Poincaré boundary problem of determining a regular solution of

$$\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}+a(x,y)\frac{\partial u}{\partial x}+b(x,y)\frac{\partial u}{\partial y}+c(x,y)u=0 \tag{1}$$

in a multiply connected domain T of smooth boundary L, such that $u(x,y)$ and its first order partial derivatives remain continuous on $T+L$ and satisfy

$$A(s)\partial u/\partial x+B(s)\partial u/\partial y+C(s)u=D(s)$$

on L, is solved by writing solutions of (1) in terms of analytic functions of a complex variable and reducing the solution of (1) to the solution of an integral equation [cf. B. V. Chvedelidze, Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 2, 571–578, 865–872 (1941); these Rev. 5, 241]. *E. F. Beckenbach.*

***Fourier, Joseph. The Analytical Theory of Heat.** G. E. Stechert and Co., New York, 1945. xxiii+466 pp. $5.00.

Photographic reprint of the English translation by Alexander Freeman. The original was published by the Cambridge University Press, Cambridge, England, in 1878.

Whitehead, S. An approximate method for calculating heat flow in an infinite medium heated by a cylinder. Proc. Phys. Soc. 56, 357–366 (1944). [MF 11535]

For small values of t a first approximation to the rapid heating or natural cooling of a cylinder of radius a of high conductivity inserted in a conducting medium may be obtained by use of the plane equation (1) $u_{xx}=D^{-1}u_t$ instead of the cylindrical equation $\theta_{rr}+r^{-1}\theta_r=D^{-1}\theta_t$; here θ is the temperature at distance r from the axis of the cylinder, $x=r-a$, $u=(r/a)^{\frac{1}{2}}\theta$, and D is the diffusivity of the medium. The accuracy of this approximation is characterized by $\frac{1}{8}Dta^{-2}$.

The solution of (1) is expressed in terms of the error function. If the specific heat of the cylinder does not exceed that of the surrounding medium, the argument of the error function is real; in the other case the argument is complex. A chart of the error function with complex argument and approximate formulae for this function with small or large argument are included in the paper.

Of the two corrections [p. 363, footnote] to E. T. Whittaker and G. N. Watson's "Modern Analysis" [4th ed., Cambridge University Press, Cambridge, England, 1927], the first [referring to p. 340, second paragraph, of that work] is erroneous [and contains a misprint]. *A. Erdélyi.*

Kostitzin, V. A. Sur l'équation de la chaleur dans le cas d'une sphère hétérogène, en tenant compte de sources et de discontinuités. Bull. Soc. Math. France 70, 125–142 (1942). [MF 11892]

Let r, ϕ, ψ be spherical coordinates and write the heat equations

$$r^2\gamma(r)\frac{\partial u}{\partial t}=\frac{\partial}{\partial r}\left(r^2K_i(r)\frac{\partial u}{\partial r}\right)+\frac{K_i}{\sin\phi}\frac{\partial^2u}{\partial\psi^2}+\frac{K_i}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right)+M(r,\phi,\psi,t), \tag{1}$$

$$r_{i-1}<r<r_i,\ i=1,\cdots,m;\ r_0=0,\ r_m=R.$$

The author gives a formal representation of a function $u(r, \phi, \psi, t)$ satisfying equations (1) and the initial condition $u(r, \phi, \psi, 0)=\omega(r, \phi, \psi)$, and also meeting boundary conditions, which are linear in u and $\partial u/\partial r$, at the spherical surfaces $r=r_j$ ($j=1, \cdots, m$). The solution is represented in the form

$$u=\sum\sum u_{nk}(r, t)S_{nk}(\phi, \psi),$$

where the S_{nk} are spherical harmonics and the u_{nk} involve orthogonal functions. The paper closes with a discussion of the temperature conditions of the earth. *F. G. Dressel.*

Bell, R. P. A problem of heat conduction with spherical symmetry. Proc. Phys. Soc. 57, 45–48 (1945). [MF 11959]

Consider a sphere of radius a concentric with a sphere of radius b ($a<b$); let the index 1 refer to the space $0<r<a$, and index 2 to $a<r<b$. The paper gives the solutions of the following heat conduction problems:

$$\partial u_i/\partial t=k_i\partial^2 u_i/\partial r^2;\quad u_1(a, t)=u_2(a, t);$$

$$K_1(a\partial u_1/\partial r-u_1)_{r=a}=K_2(a\partial u_2/\partial r-u_2)_{r=a};$$

$u_1(0, t)=u_2(b, t)=0$; $u_1(r, 0)=v_0 r$, and either $u_2(r, 0)=v_0 r$ or $u_2(r, 0)=0$. Here k_i, K_i, v_0 are constants. The problem with the initial condition $u_2(r, 0)=v_0 r$ was solved by Carslaw [Proc. Cambridge Philos. Soc. 20, 399–410 (1920)]. The author reprints the answer since he remarks that Carslaw's results are incorrect. [This must refer to a misprint in formula (33) as Carslaw has the correct answer in his formula (10).] The paper closes with some simple approximate solutions of the above problems.
F. G. Dressel (Durham, N. C.).

Ludwig, Konrad. Wärmeausgleichsvorgänge in bestrahlten Platten. Z. Angew. Math. Mech. 23, 259–269 (1943). [MF 11739]

A problem in the measurement of radiation leads to the following heat equation problem:

$$\partial\theta/\partial t=a\partial^2\theta/\partial x^2,\quad -\lambda(\partial\theta/\partial x)_{x=0}=-a_1(\theta)_{x=0},$$

$$-\lambda(\partial\theta/\partial x)_{x=\delta}=-q+a_2(\theta)_{x=\delta},\quad (\theta)_{t=0}=0,$$

where a, a_1, a_2, λ and q are constants. The problem is solved first by the classical Bernoulli method and then by the method of the Laplace transform with the use of the Mellin inversion formula. The solution is discussed in detail for values of the constants which occur for various materials and situations. *E. H. Rothe* (Ann Arbor, Mich.).

Ludwig, Konrad. Das Aufheizen einer Wand durch konstante Wärmestromdichten. Z. Angew. Math. Mech. 23, 358–360 (1943). [MF 11745]

The problem is to find a solution θ of the heat equation $\partial\theta/\partial t=a\partial^2\theta/\partial x^2$ which is zero for $t=0$ and satisfies the boundary conditions

$$-\lambda(\partial\theta/\partial x)_{x=0}=q,\quad \lambda(\partial\theta/\partial x)_{x=\delta}=q,$$

where a, q and λ are constants. The problem is solved by setting $\theta=\theta_p+\theta_0$, where θ_p is a particular solution, satisfying the boundary conditions, which is assumed to be linear in t (and therefore of degree 3 in x), while θ_0 is found by the Bernoulli method. The problem is also treated by the use of the Laplace transform. A numerical example is given. *E. H. Rothe* (Ann Arbor, Mich.).

Brandt, W. H. Solution of the diffusion equation applicable to the edgewise growth of pearlite. J. Appl. Phys. 16, 139–146 (1945).

This paper is concerned with the equation $u_{xx}+u_{yy}+ku=0$, which is derived from the equation of diffusion $u_{\xi\xi}+u_{\eta\eta}=u_t$ by making the substitution $x=\xi-kt$, $y=\eta$ and considering the steady state $t\to\infty$. The equation is treated by the conventional method of particular solutions.
W. Feller (Ithaca, N. Y.).

Michal, A. D. Studies on geodesics in vibrations of elastic beams. Proc. Nat. Acad. Sci. U.S.A. 31, 38–43 (1945). [MF 11779]

The author considers the fourth order partial differential equation for the transverse vibrations of a homogeneous elastic beam which is simply supported at both ends. By introducing a parameter s in place of the time t, it is shown that the displacement function u satisfies an integro-differential equation. Conversely, it is stated that if u satisfies the integro-differential equation then by a change in parameter u becomes a solution of the original equation subject to the original boundary conditions. The solutions u are said to form "dynamical paths." The element of arc ds for the infinite dimensional "Riemannian" space in which the trajectories are imbedded is determined. It is stated that the generalized Euler-Lagrange equation (for determining the geodesics of the Riemannian space) is the integro-differential equation previously determined. Hence the "dynamical paths" coincide with the geodesics of the Riemannian space. The solutions of the original partial differential equation which are the products of fundamental solutions represent closed geodesics which are orthogonal at the origin of the Riemannian space. *N. Coburn.*

Functional Analysis

Smiley, Malcolm. An extension of metric distributive lattices with an application in general analysis. Trans. Amer. Math. Soc. 56, 435–447 (1944). [MF 11489]

The author first shows that MacNeille's embedding (by hypercomplex algebra) of any distributive lattice L in a Boolean algebra B will also embed a metric L in a metric B. If the measure function is continuous in L, he shows that L can be embedded in a field of sets with a completely additive measure. He then extends the construction to "quasi-distributive" lattices and applies the result to show that an abstract generalization by E. H. Moore of Radon's generalization of Hellinger integration is effectively equivalent to Radon's in the case of finite total measure (bounded measure). *G. Birkhoff* (Cambridge, Mass.).

Dieudonné, J. La dualité dans les espaces vectoriels topologiques. Ann. Sci. École Norm. Sup. (3) 59, 107–139 (1942). [MF 11880]

This paper considers continuous additive transformations between topological vector spaces, in which the topology is determined by semi-norms. The author supposes the existence of a functional $B(x', x)$ additive and continuous in each variable x', x from the respective spaces E', E. For a given x', $|B(x', x)|$ is a semi-norm for x and this system of semi-norms determines a topology, the "weak" topology. The bilinear functional B also permits one to introduce the notion of orthogonality and various associated ideas. The author considers the notions of subset and quotient space relative to the weak topology. For four spaces E, E', F, F',

the author then develops the theory of weakly continuous transformations from E to F. The equivalent of the notion of adjoint is introduced in a symmetric fashion and the orthogonality relation between the range of the adjoint and the set of zeros of the given transformation is established, as well as the dual relationship for the range of the transformation itself. If the transformation is considered on the quotient space, relative to the set of zeros, the closure of the range implies the existence of a continuous inverse. These results, of course, lead to the theory of the solutions of linear operator equations. The author then discusses the specialization of his results to normed spaces, where, of course, they are known. He states explicitly the theorem that a locally convex space is normable if and only if the origin has a weakly bounded neighborhood, a result which he points out is due essentially to J. V. Wehausen.

F. J. Murray (New York, N. Y.).

Krein, M. On Hermitian operators with deficiency indices equal to one. II. C. R. (Doklady) Acad. Sci. URSS (N.S.) **44**, 131–134 (1944). [MF 12030]

Using the same notation as in part I [same C. R. (N.S.) **43**, 323–326 (1944); these Rev. **6**, 131], this note concerns the correspondence between the elements f of Hilbert space $\mathfrak{H}$ and the space $\mathfrak{H}_u$ of functions $f_u(z)$ defined by the condition that $f-f_u(z)u$ belongs to the class $\mathfrak{M}_z$ of elements of $\mathfrak{H}$ of the form $Ag-zg$, where u is a fixed element of $\mathfrak{H}$ which does not belong to $\mathfrak{M}_z$ for some z of both the upper and lower half of the z-plane. The results announced deal with the subset $\mathfrak{R}_u$ of $\mathfrak{H}_u$ of functions $f_u(z)$ which are holomorphic on the whole real axis. For these we have

$$(f, g) = \int_{-\infty}^{\infty} f_u(t)\bar{g}_u(t)d(E_t u, u) = \int_{-\infty}^{\infty} f_u(t)\bar{g}_u(t)d\sigma(t)$$

for all spectral functions E_t of A [see Livshitz, same C. R. (N.S.) **44**, 3–7 (1944); these Rev. **6**, 131]. If $\mathfrak{R}_u$ is dense in $\mathfrak{H}_u$ and in $L_\sigma^{(2)}$ we have an isometric correspondence between $L_\sigma^{(2)}$ and $\mathfrak{H}$ in which the operator of multiplying by the argument in $L_\sigma^{(2)}$ induces a self-adjoint extension of the operator A. If a is a u-regular point of A, that is, a point in some neighborhood of which all functions $f_u(z)$ of $\mathfrak{H}_u$ are holomorphic, then, for fixed z, $f_u(z)$ is a linear functional in $\mathfrak{H}$ whose norm is $\{\sum_k |\varphi_k(z)|^2\}^{\frac{1}{2}} = D(z)$, where $\varphi_k(z)$ are the functions in $\mathfrak{H}_u$ corresponding to a complete orthonormal system φ_k in $\mathfrak{H}$, the series being uniformly convergent on any closed set of u-regular points. The property of regularity of an operator (that the set $\mathfrak{M}_z = (A-zE)\mathfrak{D}_A$ is closed for all z in the complex plane) is equivalent to: (a) one of the self-adjoint extensions $\tilde{A}$ of A has an inverse which is a continuous operator; (b) for a suitable scale u, $\mathfrak{R}_u$ agrees with $\mathfrak{H}_u$. If u is such a scale for a regular A, then an $f(z)$ meromorphic in the z-plane belongs to $\mathfrak{H}_u$ if and only if there exists a C such that $|f(z)|^2 < CD(z)$, and, for some orthogonal spectral E_t, $\int_{-\infty}^{\infty} |f(t)|^2 d(E_t u, u) < \infty$.

T. H. Hildebrandt (Ann Arbor, Mich.).

Shin, D. Quasi-differential operators in Hilbert space. Rec. Math. [Mat. Sbornik] N.S. **13(55)**, 39–70 (1943). (Russian. English summary) [MF 11645]

Operators are defined in $L_2 = L_2(a, b)$ as follows:

$$f^{[0]} = p_{0,0}f, \quad f^{[k]} = ip_{kk}df^{[k-1]}/dx + \sum_0^{k-1} p_{kj}f^{[j]},$$

$$g^{[0]} = \bar{p}_{n,n}g, \quad g^{[k]} = i\bar{p}_{n-k,n-k}dg^{[k-1]}/dx + \sum_0^{k-1} q_j g^{[j]},$$

where $q_j = \bar{p}_{n-j,n-k}\bar{p}_{n-k,n-k}/\bar{p}_{n-j,n-j}$, $k=1, \cdots, n$, and the p_{kj} are complex-valued measurable functions; D_T and D_S are classes of functions f, g so that $f^{[k]}$, $g^{[k]}$ are absolutely continuous and $f^{[n]}$, $g^{[n]}$ are L_2; $Tf = f^{[n]}$, $Sg = g^{[n]}$. One has $(Tf, g)-(f, Sg) = i[f, g]_b - i[f, g]_a = i[f, g]$, with $[f, g]_x = \sum_1^n f^{[n-k]}(x)g^{[k-1]}(x)$. Let D_H (D_N) be the subclasses of functions f (g) in D_T (D_S) for which $[f, g]=0$ for all g in D_S (f in D_T); let $Hf = f^{[n]}$ $(Ng = g^{[n]})$ be an operator defined in D_H (D_N). It is proved that, if the p_{kj} satisfy the self-adjoint conditions for the $f^{[k]}$, then $T\equiv S$, $H\equiv N$ and H is symmetric. The indices (p, q) of H are given. An element f of D_T is in D_H if and only if $[f, U_k]=0$, $[f, V_j]=0$, where the U_k, V_j are orthonormalized solutions of $TU-lU=0$, $TV-\bar{l}V=0$, $\mathfrak{J}(l)\neq 0$. Conditions securing "regularity" of end points are given, as well as conditions under which f of D_T is in the domain of "maximal symmetric extension" $H^m\{a_{kj}\}$ of H (unitary matrix $\|a_{kj}\|$). In the case of regular endpoints there is a correspondence between self-adjoint operators and certain self-adjoint quasi-differential systems. The resolvent of $H^m\{a_{kj}\}$ is an integral operator with its kernel in L_2. If $p=q$, Carleman kernels are introduced. When $p=q$, $\Delta=(\alpha, \beta)$, let $E(\lambda)$ be the resolution of the spectral function corresponding to the self-adjoint $H^m\{a_{kj}\}$; then $E(\Delta)=E(\beta)-E(\alpha)$ is an operator with a Carleman kernel. If $p=q=n$ there is no continuous spectrum; if $p=q=0$, then H has no point spectrum. The work constitutes essentially an application of operator theory in Hilbert spaces to quasi-differential equations of nth order in the way this was done by M. H. Stone for second order equations.

W. J. Trjitzinsky (Urbana, Ill.).

Markouchevitch, A. Sur la meilleure approximation. C. R. (Doklady) Acad. Sci. URSS (N.S.) **44**, 262–264 (1944). [MF 12039]

Let E be a Banach space, $\{u_n\}$ a sequence of linearly independent elements of E, U the closed linear manifold determined by $\{u_n\}$. If $u \epsilon U$, the best approximation $E_n(u)$ to u of order n is defined by

$$E_n(u) = \inf \left\| u - \sum_{k=0}^{n} a_k u_k \right\|,$$

where the lower bound refers to all sequences of numbers $a_0, \cdots, a_n$. If $\{\epsilon_n\}$ is a decreasing sequence of positive numbers, any $u \epsilon U$ can be written in the form $u = u^{(1)} + u^{(2)}$, where

$$\liminf_{n\to\infty} \epsilon_n^{-1} E_n(u^{(i)}) = 0, \qquad i = 1, 2.$$

Example: every continuous function is the sum of two functions which are quasianalytic in the sense of S. Bernstein. If the u_n are in addition strongly linearly independent [Markouchevitch, same C. R. (N.S.) **41**, 227–229 (1943); these Rev. **6**, 69], in which case there is a biorthogonal system of linear functionals L_n, the author gives the following result. Let M_n be the norm of the linear operator Λ defined by $\Lambda(u) = \sum_{j=0}^n L_j(u) \cdot u_j$, $u \epsilon U$. Then, if $M_n E_n(u) \to 0$, $\sum_0^\infty L_j(u) \cdot u_j = u$; if $\liminf M_n E_n(u) = 0$, the series has a sequence of partial sums which converge to u. The M_n generalize the Lebesgue constants for Fourier series. A consequence of these theorems is the result that every $u \epsilon U$ can be written in the form $u = u^{(1)} + u^{(2)}$, where each of the series $\sum L_n(u^{(i)}) \cdot u_n$ has a subsequence of partial sums converging to $u^{(i)}$ $(i=1, 2)$; this was stated in the author's paper quoted above. (The author actually states his theorems for spaces more general than Banach spaces.)

R. P. Boas, Jr. (Providence, R. I.).

Sheffer, I. M. An extension of a Perron system of linear equations in infinitely many unknowns. Amer. J. Math. 67, 123–140 (1945). [MF 11928]

The author replaces the system of equations considered by Perron [Math. Ann. 84, 1–15 (1921)]:

$$\sum_{s=0}^{\infty} (a_s+b_{n,\,n+s})x_{n+s}=c_n,$$

in which the nth equation begins with x_n, by

$$\sum_{s=0}^{n-1} b_{ns}x_s+\sum_{s=0}^{\infty} (a_s+b_{n,\,n+s})x_{n+s}=c_n.$$

The sequence c_n belongs to the space in which

$$((c_n))=\limsup |c_n|^{1/n}<\infty,$$

and solutions x_n are sought belonging to the same space. Perron obtained an existence theorem in the case where (1) a_s are such that $F(t)=\sum_{s=0}^{\infty}a_st^s$ is analytic in $|t|\leqq 1$, (2) $|b_{n,\,n+s}|\leqq k_n\theta^s$, where $\theta<1$ and $k_n\rightarrow 0$, and (3) $a_0+b_{nn}\neq 0$ for all n. The condition (3) is discarded, while the initial terms are subjected to the condition (4) that there exists a ρ, $0\leqq\rho<1$, such that $((\sum_{s=0}^{n-1}b_{ns}k_s))\leqq\rho$ for all x_n such that $((x_n))\leqq 1$. Under these conditions it is shown that, if $((c_n))\leqq 1$ and $F(t)$ has λ zeros in $|t|\leqq 1$, there exist a number G and an integer $M\geqq 0$ independent of c_n such that the system obtained by deleting the first M equations has a unique solution x_n with $((x_n))\leqq G$ and preassigned values $x_0, \cdots, x_{\lambda+M-1}$. Conditions are determined under which for $G=1$ this solution can be extended to the entire system, the values of x_n, $0\leqq n\leqq\lambda+M-1$, being preassigned.

T. H. Hildebrandt (Ann Arbor, Mich.).

Calculus of Variations

Morrey, Charles B., Jr. Multiple integral problems in the calculus of variations and related topics. Univ. California Publ. Math. (N.S.) 1, 1–130 (1943). [MF 10556]

The present memoir is concerned with the following two problems: first, to establish existence theorems for the problem of minimizing a multiple integral

$$I(z, G)=\int_G f(x, z, p)dx$$
$$=\int_G f(x_1, \cdots, x_n, z_1, \cdots, z_m, p_1{}^1, \cdots, p_m{}^n)dx_1\cdots dx_n$$

over a class F of functions $z:z_j(x)$ (x on G, $j=1, \cdots, m$); second, to show that under certain conditions a minimizing function $z(x)$ possesses additional differentiability properties. These two problems are very difficult and the author makes important contributions towards their solution. In order to obtain existence theorems the class of admissible functions z is enlarged so as to include functions $z(x)$ that are not necessarily continuous but possess partial derivatives in a generalized sense. As is to be expected, the hypotheses on the integrand f are more stringent than those commonly used for the case when $n=1$. The results obtained are analogous to those obtained by Tonelli for double integrals [Ann. Scuola Norm. Super. Pisa (2) 2, 89–130 (1933)]. The problem of Plateau, together with certain generalizations, is treated separately.

In regard to differentiability theorems, the author restricts himself to the case $n=2$ and makes certain additional assumptions regarding the integrand f. The case considered is sufficiently general to yield significant extensions of results obtained heretofore. For example, it is shown that under very general hypotheses the first derivatives of a minimizing function z satisfy a uniform Hölder condition. Additional assumptions imply that z has derivatives of higher order.

The paper is divided into seven chapters. The first two deal chiefly with a description of the properties of the class of functions to be considered. Free use is made of results of Calkin and the author [Duke Math. J. 6, 170–186; 187–215 (1940); these Rev. 1, 208]. The third chapter is concerned with existence theorems. Lemmas in potential theory are derived in chapter IV. These lemmas are of interest apart from their applications to the theory of differential equations and calculus of variations. A general form of Haar's lemma is treated in chapter V. The Haar equations are written in the integral form given by Coral [Duke Math. J. 3, 585–592 (1937)]. A study of the solutions of elliptic systems of linear self-adjoint partial differential equations in two independent variables is found in chapter VI. These equations are taken in the integral form suggested by the Haar-Coral equations. Further study of systems of this type should be fruitful. Differentiability theorems are obtained in chapter VII. *M. R. Hestenes* (Chicago, Ill.).

Tonelli, Leonida. Su un nuova tipo di problemi di calcolo delle variazioni. Ann. Scuola Norm. Super. Pisa (2) 10, 23 pp. (1941) = Ist. Naz. Appl. Calcolo (2) no. 123. [MF 11728]

Consider a rectangle R defined by $X_0\leqq x\leqq X_1$, $Y_0\leqq y\leqq Y_1$, a value Y_i such that $Y_0\leqq Y_i<Y_1$, a positive constant a and two continuous functions $\phi(y)$ and $\psi(y)$ on (Y_0, Y_1). It is furthermore supposed that $\phi'(y)$ and $\phi''(y)$ are continuous and that ϕ and ψ are such that $\phi(Y_1)=0$, $\phi(y)>0$ on $(Y_0\leqq y<Y_1)$, $\psi(Y_1)<0$. The author proposes the following interesting problem in the calculus of variations: to minimize the integral

$$\int_{X_0}^{X_1}[ay'(x)+(a^2y'^2(x)+\phi(y(x)))^{\frac{1}{2}}]dx/\phi(y(x))$$

in the class of all absolutely continuous functions $y(x)$ on (X_0, X_1) such that $y(X_0)=Y_i$, $Y_0\leqq y(x)\leqq Y_1$ and $y'(x)\geqq\psi(y(x))$ on (X_0, X_1), and such that the integrand is Lebesgue integrable. The author points out that this is a special case of another new problem. Let $F(y, x', y')$ be real-valued, defined for $Y_0\leqq y\leqq Y_1$ and for all pairs $(x', y')\neq(0, 0)$, and let F be continuous along with its first and second partial derivatives for $Y_0\leqq y<Y_1$; moreover, assume that $F(y, x', y')>0$ on $Y_0\leqq y<Y_1$ provided $(x', y')\neq(0, r)$ with $r<0$. It is furthermore supposed that F is positively homogeneous of degree 1 in (x', y'), that the Weierstrass invariant $F_1=F_{x'x'}/y'^2$ is positive and that $F_{x'}(y, 0, y')=0$ for $Y_0\leqq y<Y_1$, $y'\neq 0$. Finally, it is assumed that there is a $\gamma>0$ such that $F(y, x', y')>\gamma y'$ for $(x', y')\neq(0, 0)$, $x'\geqq 0$ and $Y_0\leqq y<Y_1$. Let $f(y, y')=F(y, 1, y')$ and consider the problem of minimizing the integral

$$\int_{X_0}^{X_1} f(y(x), y'(x))dx/\phi(y(x))$$

in the class K of all absolutely continuous functions on (X_0, X_1) for which $y(X_0)=Y_i$, $Y_0\leqq y(x)\leqq Y_1$, $y'(x)\geqq\psi(y(x))$ and for which $f(y(x), y'(x))/\phi(y(x))$ is Lebesgue integrable.

The author shows that K is not empty and then enlarges his class of admissible arcs to include certain arcs representable in parametric form as $x=x(s)$, $y=y(s)$ with s the arc length. He then proves that his integral has an absolute minimum in the enlarged class K^* of admissible arcs and next shows that a minimizing arc is in the original class K. The class K^* consists of all continuous and rectifiable curves C lying in a rectangle R^*: $X_0 \leqq x \leqq X_1$, $Y_0 \leqq y \leqq Y^*$, where $Y^* < Y_1$, passing through (X_0, Y_i) and having their second end-point on the line $x = X_1$, having $x(s)$ a nondecreasing function of arc length, and having the tangent of the angle they make with the x-axis not less than $\psi(y)$ almost everywhere. The existence proof is made by showing that the minimizing sequence of arcs are uniformly bounded in length, all lie in R^* and hence have an accumulation arc. The integral is then shown to be lower semi-continuous. The paper closes with a proof that the minimizing arc is actually in the original class K. *H. H. Goldstine.*

TOPOLOGY

Eckmann, B. Topologie und Algebra. Vierteljschr. Naturforsch. Ges. Zürich **89**, 25–34 (1944). [MF 11537]

Lecture discussing the concepts of space and number (geometry and algebra), their apparent contrast, their relationships as exemplified by synthetic and analytic projective geometry and by combinatorial ("algebraic") topology, and finally their synthesis in systems like Pontrjagin's duality theory for Abelian topological groups.

H. Samelson (Syracuse, N. Y.).

Gama, Lelio I. Notion de proximité et espaces à structure sphéroïdale. Amer. J. Math. **67**, 42–58 (1945). [MF 11921]

An accessible space (class (H) of Fréchet) is regularly accessible provided that to each point a there corresponds a sequence of neighborhoods $V_1(a) \supset \cdots \supset V_n(a) \supset \cdots$ almost all of which are contained in any neighborhood of a; it is strictly accessible if Hausdorff topological; finally, it is uniformly accessible if for each $V(a)$ there exist $V' \subset V$ and an integer n' such that $a \epsilon V'$ and $n > n'$ imply $V_n(a) \subset V$. Two points a,b of a regularly accessible space are called near (proches) if there exists an integer n such that $a \epsilon V_n(b)$ and $b \epsilon V_n(a)$. The upper bound of n for which these conditions are satisfied (denoted by $\pi(a,b)$) is the order of nearness or proximity of points a,b. If a,b are near, $\pi(a,b)$ is a positive integer; if a,b are not near, $\pi(a,b)$ is defined to be zero; if $a=b$, $\pi(a,b)=\infty$. The author utilizes this notion of proximity in studying the general topology of accessible spaces, first applying it to the theory of sequences of elements in uniformly accessible spaces. A regularly accessible space is spheroidal if corresponding to each integer N there is an integer ν such that for any two points a,b of the space (distinct or not) $V(a)\cdot V(b) \neq 0$ and $\alpha, \beta \epsilon (V_\nu(a) + V_\nu(b))$ imply $\pi(\alpha,\beta) \geqq N$. The author lists some of the fundamental properties of spheroidal spaces and proves that a regularly accessible space is metric if and only if it is spheroidal. Seeking to study metric spaces without appealing to the notion of distance, the author shows how the concept of proximity may be used to obtain the normality property and the Borel-Lebesgue theorem in spheroidal spaces. This program is continued in the remaining three sections of the paper which are devoted to sequences of sets, Cantorian sets and transitive properties. (A property P of a set is called transitive provided that if $E(\neq 0)$ has property P and $E=E_1+E_2$ then E_1 or E_2 has property P.) *L. M. Blumenthal* (Columbia, Mo.).

White, Paul A. Additive properties of compact sets. Duke Math. J. **11**, 699–701 (1944). [MF 11572]

The author considers a general property p^i, concerned with i-cycles, of subsets of a compact metric space. All the complexes and cycles are nonoriented and the Vietoris cycles used consist of these cycles as coordinates. The property p^i is said to be additive provided that, if M_1 and M_2 are two compact subsets, having p^i, of a compactum M_1+M_2, and if $M_1\cdot M_2$ has p^{i-1}, then M_1+M_2 has p^i. It is shown that the following properties are additive: (uniform) local i-connectedness; (uniform) simple i-connectedness; the property of being (uniformly) locally i-connected and at the same time containing (uniformly) all non-i-cut points; the property of being simply i-connected and at the same time containing all non-i-cut points; (uniform) semi-i-connectedness if the product set is required to have the stronger property of (uniform) local-$(i-1)$-connectedness (which implies semi$(i-1)$-connectedness). Let $P(n, j)$ be the property of being a compact simply i-connected $(i=0, 1, \cdots, n-2)$ lc^{n-2} embedded in E_n, having only simply-j-connected $(j \leqq n-2)$ generalized closed $(n-1)$-manifolds [see R. L. Wilder, Duke Math. J. 1, 543–555 (1935)] as boundaries of its complementary domains. Then $P(n, j)$ is additive with respect to n and j together; that is, if A and B have $P(n, j)$ and $A\cdot B$ has $P(n-1, j-1)$, then $A+B$ has $P(n, j)$. *D. W. Hall.*

Moore, R. L. Concerning tangents to continua in the plane. Proc. Nat. Acad. Sci. U.S.A. **31**, 67–70 (1945). [MF 11952]

A line L (straight line ray pb) is said to be tangent to a nondegenerate plane continuum M at $p \epsilon M$ provided every subset of M having p as a limit point intersects every domain which contains $L-p$ and is composed of two vertical angles with vertex at p (contains b and is the interior of an angle with vertex at p). It is shown first that no continuum M contains uncountably many points each of which is the origin of a ray tangent to M at that point. Furthermore, if a compact continuum M has a tangent at each of its points, then (1) every subcontinuum of M is a continuous curve (compactness is not needed for this), (2) the set K of all emanation points of triods lying in M has a totally disconnected closure and (3) the set of all $p \epsilon M$ such that p is the origin of a ray tangent to M at p is a countable inner limiting set and its closure is totally disconnected. Conversely, any countable compact set K with a totally disconnected closure is the set of all junction points of some dendron M which is topologically equivalent to a continuum having a tangent at every one of its points.

G. T. Whyburn (Charlottesville, Va.).

Shanks, M. E. Monotone decompositions of continua. Amer. J. Math. **67**, 99–108 (1945). [MF 11926]

The author is concerned with monotone upper semi-continuous decompositions of a T_1 space. Let $\mathfrak{D}_1(X)$ be the class of all decompositions of a compactum X into disjoint closed sets; $\mathfrak{D}_1$ is then partially ordered by saying that $D_1 < D_2$ if each subset of D_2 is in some subset of D_1. The class $\mathfrak{D}_m(X)$ is the class of monotone decompositions (those whose subsets are continua); $\mathfrak{D}_{sm}(X)$ is the class of simple monotone decompositions which are upper semi-

continuous (a decomposition is simple if it has at most one subset consisting of a single point). Then it is shown that two continua X and Y are homeomorphic if and only if there is an isomorphism of $\mathfrak{D}_m(X)$ onto $\mathfrak{D}_m(Y)$ carrying $\mathfrak{D}_{sm}(X)$ onto $\mathfrak{D}_{sm}(Y)$. If moreover X and Y are also locally connected, they are homeomorphic if and only if $\mathfrak{D}_{sm}(X)$ is isomorphic to $\mathfrak{D}_{sm}(Y)$. In the second section the author shows that, if X, X_1, X_2 are compacta with $X=X_1+X_2$ and if $X_1\cdot X_2$ is a finite set of points, then $\mathfrak{D}_m(X)$ is isomorphic to the direct product of $\mathfrak{D}_m(X_1)$ and $\mathfrak{D}_m(X_2)$. If A is an arc and X a continuum such that $\mathfrak{D}_m(X)$ is isomorphic to $\mathfrak{D}_m(A)$, then X is hereditarily locally connected. Finally, if X is a linear graph or a dendrite, then $\mathfrak{D}_m(X)$ is isomorphic to the $\mathfrak{D}_m(A)$ of an arc A. In the third section a generalized dendrite is defined as a continuum X having a unique irreducible subcontinuum containing every pair of its points. A necessary and sufficient condition that a continuum X be a generalized dendrite is for $\mathfrak{D}_m(X)$ to be a sublattice of $\mathfrak{D}(X)$, where $\mathfrak{D}(X)$ is the set of all upper semi-continuous decompositions. The join $D_1\vee D_2$ is the decomposition whose subsets are the intersections of subsets of D_1 and D_2; $D_1\wedge D_2$ is the join of all decompositions less than both. If X is a Knaster continuum or if X is a dendrite, then $\mathfrak{D}_m(X)$ is a distributive sublattice. Finally a continuum X is a dendrite if and only if $\mathfrak{D}_m(X)$ is a sublattice of $\mathfrak{D}(X)$ such that $\mathfrak{D}_m(X)$ is isomorphic to $\mathfrak{D}_m(A)$, with A an arc or such that simple monotone decompositions have their complements in $\mathfrak{D}_m(X)$. *H. H. Goldstine.*

Whyburn, G. T. Coherent and saturated collections. Trans. Amer. Math. Soc. **57**, 287–298 (1945). [MF 12135]

The space M is separable metric and G is a collection of subsets of M. Two points of M are (G, α)-conjugate if they are not separated in M by α sets of the collection G. The author begins with a series of general results involving decompositions of M by means of (G, α)-conjugacy; this portion of the paper does not lend itself readily to a brief exposition. A collection G' of subsets of M is said to be (G, α)-determined provided that, if $x\epsilon g'\epsilon G'$, then x is (G, α)-conjugate to y if and only if $y\epsilon g'$. From the principal theorem it follows that any (G, α)-determined collection G in a continuum M generates a nonalternating transformation $f(M)=M'$. When α is permitted to vary over the finite cardinals (or is fixed at aleph-null or c) the function is monotone. Also, f is monotone if $\alpha=1$ and the sets of G are continua. It is also shown that in order that a continuum admit a monotone transformation onto an interval it is necessary and sufficient that it contain an uncountable collection of disjoint connected cuttings. An analogous proposition concerns mappings into the circle. The author gives the following elegant generalization of a theorem due to J. H. Roberts and R. L. Moore. If F is a closed set in a Peano space M then M admits a monotone transformation onto a regular curve N such that each component of F is the inverse of a point of N. *A. D. Wallace.*

White, Paul A. Regular transformations. Duke Math. J. **12**, 101–106 (1945). [MF 12073]

The mapping $T(K)=K'$ is said to be n-regular if, when $y_i\rightarrow y$ in K', then $T^{-1}(y_i)\rightarrow T^{-1}(y)$ s-regularly in K for all $s\leqq n$. Here K, K' are compact metric. Roughly, s-regular convergence means that small cycles in the converging sets bound in uniformly small sets near the limit set. After developing certain tool-theorems the author gives the following results. (1) If T is n-regular on the continuum K then T can be factored $(T=T_2T_1)$ so that T_1 is monotone and n-regular and T_2 is locally topological and interior. (2) In order that T be n-regular it is necessary and sufficient that the n-regular convergence $Y_i\rightarrow Y$ imply the n-regular convergence $T^{-1}(Y_i)\rightarrow T^{-1}(Y)$. (3) The product of n-regular mappings is n-regular. (4) If T is n-regular and $Y\subset K'$ is lc^n then $T^{-1}(Y)$ is lc^n. Many of the results are extensions of theorems due to W. T. Puckett and the reviewer. There are obvious misprints in the paper, particularly the first definition. *A. D. Wallace* (Philadelphia, Pa.).

Begle, Edward G. Duality theorems for generalized manifolds. Amer. J. Math. **67**, 59–70 (1945). [MF 11922]

The definition of an open generalized manifold is obtained from the author's previously given definition of a generalized manifold [Amer. J. Math. **64**, 553–574 (1942); these Rev. **4**, 87] by reading "locally compact" instead of "compact." The coefficient group must be a field. The Poincaré and Alexander duality theorems are extended to open generalized manifolds. *R. H. Fox* (Syracuse, N. Y.).

Pontrjagin, L. On some topologic invariants of Riemannian manifolds. C. R. (Doklady) Acad. Sci. URSS (N.S.) **43**, 91–94 (1944). [MF 11610]

In a preceding paper [C. R. (Doklady) Acad. Sci. URSS (N.S.) **35**, 34–37 (1942); these Rev. **4**, 147] the author has defined certain characteristic cocycles of a differentiable manifold M. In this paper integral formulas are derived for these cocycles. The integrands are differential forms (alternating tensors) constructed out of the curvature tensor of a metric obtained from an imbedding of M in a Euclidean space. Integral formulas for the Stiefel-Whitney cocycles (which are included among those of Pontrjagin) have also been given by S. Chern [Proc. Nat. Acad. Sci. U.S.A. **30**, 269–273 (1944); these Rev. **6**, 106]. In the procedure of Chern, the metric of M need not be derived from an imbedding in a Euclidean space. *N. E. Steenrod.*

GEOMETRY

Steck, Max. On an axiom of continuity equivalent to the weak E. P. **axiom.** Revista Mat. Hisp.-Amer. (4) **3**, 295–301 (1943). (Spanish) [MF 12155]

The weak E.P. axiom asserts that in addition to points outside and points on a nondegenerate conic K_2 there is at least one point inside K_2 (that is, at least one point such that every line through it meets the conic in two distinct points). The author has exploited this axiom in a series of notes [Monatsh. Math. Phys. **49**, 209–212 (1940); S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1939**, 269–276 (1939); these Rev. **2**, 135; **3**, 13] the axiomatic background of which stems from the system of axioms for real projective geometry used by Liebmann [Synthetische Geometrie, Leipzig-Berlin, 1934]. The author is now concerned with showing that the weak E.P. axiom is equivalent (within the same axiom framework used in earlier papers) to the weak H.G. axiom, which asserts that with respect to every nondegenerate conic K_2 there is at least one (nonsecant) line through each point of which pass two distinct tangents to the conic. Since the weak H.G. axiom is clearly the dual

of the weak E.P. axiom when the geometry is such that the tangents to a point-conic form the dual line-conic, the equivalence of the two axioms is not surprising.
L. M. Blumenthal (Columbia, Mo.).

Topel, Bernard J. Bolyai-Lobachevsky planes with finite lines. Rep. Math. Colloquium (2) 5–6, 40–42 (1944). [MF 11405]

In Jenks's postulates for plane hyperbolic geometry it is assumed that each line contains at least five distinct points. Here it is proved that if a plane has finite lines (that is, lines containing a finite number of points) it must be homogeneous with two points on each line. In such a plane the total number of points ($\geqq 5$) is arbitrary; the n-dimensional simplex is given as a model. *G. de B. Robinson.*

Perron, Oskar. Neuer Aufbau der nichteuklidischen (hyperbolischen) Trigonometrie. Math. Ann. 119, 247–265 (1944). [MF 11899]

This paper is probably the "purest" treatment of hyperbolic trigonometry that has ever been devised. It involves only points and lines in the ordinary hyperbolic plane, whereas Bolyai and Lobachevsky used horospheres, Liebmann (while remaining in the plane) still used horocycles, and Klein embedded the hyperbolic plane in a projective plane. Even the recent treatment by Gerretsen [Nederl. Akad. Wetensch., Proc. 45, 360–366, 479–483, 559–566 (1942); these Rev. 6, 13] has not the same "purity," since it involves points at infinity.

Using classical arguments reminiscent of Saccheri and Legendre, the author considers the sides a and b of a right triangle ABC as functions of its angle $A=\alpha$ and hypotenuse c, proving that the ratios a/c and b/c have limits $S(\alpha)$ and $C(\alpha)$, respectively, as c tends to zero. He then proves that $S(\alpha)$ and $C(\alpha)$ are continuous functions of α, which satisfy

$$S(\alpha_1+\alpha_2)=S(\alpha_1)C(\alpha_2)+C(\alpha_1)S(\alpha_2),$$
$$C(\alpha_1+\alpha_2)=C(\alpha_1)C(\alpha_2)-S(\alpha_1)S(\alpha_2),$$

whence, eventually, $S(\alpha)=\sin\alpha$ and $C(\alpha)=\cos\alpha$. With the help of differential calculus, he deduces Lobachevsky's two fundamental formulas: $\Pi(x)=2\arctan e^{-x}$ for the angle of parallelism, and

$$\cosh a=\cosh b\cosh c-\sinh b\sinh c\cos A$$

for the general triangle ABC. *H. S. M. Coxeter.*

Tietze, Heinrich. Das Volumen von gewissen Polyedern. Math. Ann. 119, 221–246 (1944). [MF 11898]

Attention is confined to the region $x_i>0$ $(i=1,\cdots,n)$ of Euclidean n-space. For a given point $P:(a_1,\cdots,a_n)$, the n-dimensional box $0\leqq x_i\leqq a_i$ $(i=1,\cdots,n)$ is denoted by $k(P)$. Points $(P_1,\cdots,P_m)$ are called independent if P_i belongs to $k(P_j)$ only when $i=j$. Given m independent points $(P_1,\cdots,P_m)$, consider the problem of obtaining a formula for the n-dimensional volume of the union of the regions $k(P_j)$ $(j=1,\cdots,m)$. This problem is solved here for the special case of n points

$$P_1:(a_1^1,\cdots,a_n^n),\quad P_i:(a_1^{n-i+2},\cdots,a_{i-1}^n,a_i^1,\cdots,a_n^{n-i+1})$$

$(i=2,\cdots,n)$ in such relative positions that $a_i^p<a_i^{p+1}$ $(i=1,\cdots,n;\ p=1,\cdots,n-1)$, and for the first m $(m=1,\cdots,n-1)$ of such a set of points. Complications are revealed which arise in other cases. *S. S. Cairns.*

Sydler, J.-P. Sur la décomposition des polyèdres. Comment. Math. Helv. 16, 266–273 (1944).

Let P, P' be two polytopes in a space in which congruence is defined. Then P is said to be equivalent to P' if P and P' may be divided into sub-polytopes $p_1,\cdots,p_n;p_1',\cdots,p_n'$ such that p_i is congruent to p_i'. Furthermore, P and P' are said to be "equivalent by adjunction" if there exist two other equivalent polytopes Q, Q' such that $P+Q$ is equivalent to $P'+Q'$. The author is concerned with the special case of equivalence of polyhedra in Euclidean 3-space. His principal theorem is the following. Let P be any polyhedron, and let $a_1,\cdots,a_n$ be n positive numbers whose sum equals 1. Then there exist n polyhedra p_i, all similar to P, the ratio of corresponding sides of p_i and P being a_i, and a polyhedron R which is equivalent to a cube, such that P is equivalent to $p_1+\cdots+p_n+R$. With the aid of this result, the author deduces the following results. Two polyhedra which are equivalent by adjunction are equivalent; a necessary and sufficient condition that a polyhedron is equivalent to a cube is that it is equivalent to a finite sum of polyhedra similar to itself. The last of these theorems includes a previous result of M. Dehn [Math. Ann. 55, 465–478 (1902)] to the effect that a regular tetrahedron is not equivalent to a cube. Finally, a purely geometric proof is given of another result of Dehn [Math. Ann. 59, 84–88 (1904)] that, if equivalent polyhedra are considered as belonging to the same class, the set of such distinct classes has the power of the continuum. *S. A. Jennings.*

Tchetveroukhine, N. Sur la "dureté affine" des polyèdres. C. R. (Doklady) Acad. Sci. URSS (N.S.) 42, 3–6 (1944). [MF 11623]

A polyhedron P' is said to be affinely rigid if every isomorphic polyhedron whose faces are affinely related to the corresponding faces of P' is itself affinely related to P'. For instance, every polyhedron with trihedral vertices is affinely rigid; so also is any pyramid. A necessary and sufficient condition for affine rigidity is given in terms of a kind of projection (not very clearly described). Equation (1) might well have been simplified to $p_n=E+2$, where E is the number of edges. *H. S. M. Coxeter.*

Kelly, L. M. Covering problems. Nat. Math. Mag. 19, 123–130 (1944). [MF 11933]

An exposition of elementary covering theorems in distance geometry (in the sense of Menger and Blumenthal). The problems treated include the covering of (a) plane sets with circular disks, (b) plane sets with circles (curves), (c) three-dimensional sets with circles, (d) plane sets with equilateral hyperbolas and (e) three-dimensional sets with spherical surfaces. *C. B. Allendoerfer* (Haverford, Pa.).

Scherk, Peter. On differentiable arcs and curves. IV. On the singular points of curves of order $n+1$ in projective n-space. Ann. of Math. (2) 46, 68–82 (1945). [MF 11789]

[Parts I and II appeared in Časopis Pěst. Mat. Fys. 66, 165–171, 172–191 (1937), part III in Ann. Mat. Pura Appl. (4) 17, 291–305 (1938).] Let C: $p(s)$ be a continuous curve in real projective n-space. Points $p(s_1)=p(s_2)$ with $s_1\neq s_2$ are considered as different points of C. The osculating $L_i(s)$ of C at $p(s)$ are defined as follows. The osculating $L_0(s)$ is $p(s)$. If $L_{i-1}(s)$ exists then $L_i(s)$ is the limit of the i-dimensional hyperplane through $L_{i-1}(s)$ and $p(s+h)$ for $h>0$. If all $L_i(s)$ exist for all s then C is called differentiable.

For a given s, numbers $a_k=1, 2, k=0, 1, \cdots, n-1$, can be defined so that $a_0=1$ and $a_0+\cdots+a_i$ is odd if and only if a hyperplane L_{n-1} which contains $L_i(s)$ but not $L_{i+1}(s)$ decomposes C in the neighborhood of $p(s)$. The paper studies differentiable curves C of order $n+1$. For any $p(s)$ of such a curve at most one a_k equals 2. The point is regular if all $a_i=1$. If $a_k=2$ then $p(s)$ is counted as a singular point of multiplicity $n-k$. This agrees with the definitions of algebraic geometry. It is shown that the sum of the multiplicities of the singular points is at most $n+1$ and is congruent to $n+1$ (mod 2). With a proper definition of intersection multiplicities there is for fixed n only a bounded number of s such that the osculating $L_{n-2}(s)$ intersect the curve n times. Some special types of curves of order $n+1$ are discussed. *H. Busemann* (Chicago, Ill.).

Wunderlich, Walter. Zur Reflexion von Röntgenstrahlen an Kristallen. Z. Phys. 122, 86–97 (1944). [MF 11811]

Es wird die bei der Reflexion eines monochromatischen Röntgenstrahlbündels an der ebenen Oberfläche eines Einkristalls entstehende Regelfläche untersucht und ihre wichtigsten ebenen Schnitte, die als Diagrammkurven auftreten können, werden gekennzeichnet. Ferner wird die Auswertung besonderer Diagrammkurven besprochen.

Author's summary.

Graf, Ulrich. Affine Transformationen durch Doppel-Photographie. Z. Angew. Math. Mech. 23, 230–236 (1943). [MF 11751]

This paper discusses methods of obtaining an affine distortion of a plane drawing by means of two successive photographs each of which constitutes a projective transformation. The photographs are assumed to be exact perspective pictures. *O. Neugebauer* (Providence, R. I.).

Thébault, Victor. Contribution à la géométrie du triangle. C. R. Acad. Sci. Paris 218, 433–435 (1944). [MF 12113]

Sen, D. K. Pedal equations of a circle. J. Univ. Bombay (N.S.) 9, part 3, 11–15 (1940). [MF 12188]

Werkmeister, P. Bestimmung der Koordinaten der Schnittpunkte einer Geraden mit einem Kreis. Z. Angew. Math. Mech. 23, 296–299 (1943). [MF 11749]

Convex Domains, Integral Geometry

Moran, P. A. P. Measuring the surface area of a convex body. Ann. of Math. (2) 45, 793–799 (1944). [MF 11383]

The author shows how the surface area of a small convex solid may be evaluated, with an accuracy sufficient for practical purposes, by projecting the area upon the surface of a dodecahedron or an icosahedron. Vectors are used and both a lower and an upper limit are computed for the estimated and the exact area. *N. A. Court.*

Liberman, I. Characterization of convex bodies. Rec. Math. [Mat. Sbornik] N.S. 13(55), 239–262 (1943). (Russian. English summary) [MF 11652]

[The exact translation of the Russian title is "On some characteristic properties of convex bodies."] Let F be a closed bounded set in E^n. A supporting L^i of F, $i=0, \cdots, n-1$, is an i-dimensional linear subspace of E^n which is contained in a supporting $L^{i+1}(L^n=E^n)$ and such that L^i is a supporting L^i of $L^{i+1}\cap F$ in the L^{i+1} in the usual sense; that is, L^i contains points of $L^{i+1}\cap F$ and $L^{i+1}\cap F$ lies entirely in one of the closed half spaces of L^{i+1} determined by L^i. The intersection of F with a supporting L^i is called an i-face of F. The principal theorem is as follows. If (and obviously only if) F and each $(n-1)$-face of F can be contracted to a point on itself then F is convex. The proof proceeds by induction with respect to the dimension. It is based on a detailed discussion of the supporting L^i and the i-faces of F, which yields interesting new results for the theory of convex bodies, and on the following fact. Call the closed set ϕ homologous to 0 on itself if for every $\epsilon>0$ a $\delta>0$ exists such that every (integral, homogeneous) δ-cycle in ϕ is the boundary of an ϵ-complex in ϕ. (The terminology of Alexandroff-Hopf is used: a δ-simplex is a finite set of points in ϕ such that the distance of any two points is less than δ, etc.) In order that all boundary points of the convex closure of F belong to F it is necessary and sufficient that each $(n-1)$-face of F is homologous to 0 on itself.

H. Busemann (Chicago, Ill.).

Preisig, E. Über Bewegungsmittelwerte konvexer Körper in Gittern. Comment. Math. Helv. 15, 120–143 (1943).

Let B be a convex rigid body undergoing all possible displacements in a space S of k dimensions, and L_k be a unit point lattice in S; then it is well known that the mean value of the number of points of L_k in B is equal to the volume W_0 of B. The author generalizes this by considering a unit sub-space lattice $L_{k-\lambda}$, consisting of (parallel) λ-dimensional sub-spaces, and taking the mean value $J_{k-\lambda}$ of the number of sub-spaces of $L_{k-\lambda}$ that meet B. He finds that

$$W_\lambda=\kappa_k J_{k-\lambda}/\kappa_{k-\lambda},$$

where κ_n denotes the volume of the n-dimensional unit sphere, and W_λ the λth "mixed volume" of B with a k-dimensional unit sphere. In particular, the surface area of B is $k\kappa_k J_{k-1}/\kappa_{k-1}$. *H. P. Mulholland* (Beirut).

Sinogowitz, Ulrich. Herleitung aller homogenen nicht kubischen Kugelpackungen. Z. Kristallogr., Mineral. Petrogr. Abt. A. 105, 23–52 (1943). [MF 11813]

A set of nonoverlapping spheres in ordinary space is called a "sphere-arrangement" (Kugellage) if one of the 230 space groups (of congruent transformations, with a finite fundamental region) is transitive on the spheres. The spheres are taken to be as large as possible, provided they do not intersect; thus each touches at least one other. The arrangement is called a "packing" (Kugelpackung) if it is connected. Extreme instances are the close packings, where each sphere touches twelve others, and the "thin" packings, where each touches only three others. [Cf. Heesch and Laves, same Z. Abt. A. 85, 443–453 (1933).] A sphere-arrangement is determined by the position of the center of one sphere in the fundamental region of the space group. Two space groups are considered equivalent if one is an affine transform of the other. For a given space group, the "type" of a sphere-arrangement is determined by the relation of the typical center to the symmetry elements of the group (for example, by the center's lying on a certain plane or axis of symmetry, or on a screw axis, etc.). Two sphere-arrangements are considered equivalent if one can be derived from the other by continuous motion of the typical center without altering its type.

The author explains how he enumerated, for the space groups of the triclinic, monoclinic, and orthorhombic systems, $46+729+2087=2862$ distinct sphere-arrangements, among which are 681 packings. The paper is clearly written, well illustrated and provided with an index.

H. S. M. Coxeter (Toronto, Ont.).

Algebraic Geometry

Segre, B. Arithmetic upon an algebraic surface. Bull. Amer. Math. Soc. **51**, 152–161 (1945). [MF 11832]

A statement of various theorems concerning the existence of rational points, or one- or two-parameter systems of rational points, on a rational cubic surface. Proofs are not given, but some of the methods of proof are indicated. These, as well as the theorems themselves, are mainly of an algebro-geometric nature.

R. J. Walker.

Fano, Gino. Alcune questioni sulla forma cubica dello spazio a cinque dimensioni. Comment. Math. Helv. **16**, 274–283 (1944).

This paper compares the results obtained in an earlier one by the same author [Ann. Mat. Pura Appl. (3) **10**, 251–285 (1904)] with those obtained by E. G. Togliatti [Vierteljschr. Naturforsch. Ges. Zürich **85** Beiblatt (Festschrift Rudolf Fueter), 127–132 (1940); these Rev. **3**, 15]. The first concerns the properties of a general primary in [4]. Most but not all of these can be extended to higher spaces. The other paper had for its purpose the study of the properties of a certain quintic surface in [3]. Interest in the present paper is largely didactical. The results of both previous papers are easily accessible; they are shown to agree throughout when properly interpreted.

V. Snyder.

Enriques, Federigo. Sui sistemi continui di curve appartenenti ad una superficie algebrica. Comment. Math. Helv. **15**, 227–237 (1943).

In this paper a new approach is made to the study of continuous systems of curves on algebraic surfaces and certain fundamental theorems are thereby proved "without the necessity of referring to delicate infinitesimal considerations." The author states that this method leads to more rigorous proofs of these theorems than methods used previously by himself and other mathematicians.

T. R. Hollcroft (Aurora, N. Y.).

Severi, Francesco. Intorno ai sistemi continui di curve sopra una superficie algebrica. Comment. Math. Helv. **15**, 238–248 (1943).

The author commences with a reply to comments made by F. Enriques in the paper reviewed above concerning an earlier paper [F. Severi, Mem. Accad. Naz. Italia **12**, 337–340 (1941)]. The question concerns the completeness of the characteristic series of complete systems of curves on an irreducible algebraic surface, especially when, as limit, the variable system includes a given composite curve, and whether the arguments, applied to regular systems, can be extended to superabundant ones [B. Segre, Ann. Mat. Pura Appl. (4) **17**, 107–126 (1938)]. The difficulty is caused by inadequately defined premises in both papers.

V. Snyder (Ithaca, N. Y.).

Perron, Oskar. Studien über den Vielfachheitsbegriff und den Bézoutschen Satz. Math. Z. **49**, 654–680 (1944). [MF 11977]

This paper is devoted chiefly to a rather severe criticism of some recent results of F. Severi. Given a solution of a system of k equations $f_\nu(x_1, \cdots, x_{k+1})=0$, where f_ν is a homogeneous polynomial of degree n_ν with complex coefficients, Severi [Abh. Math. Sem. Hansischen Univ. **15** (1943)] proposed two definitions for the multiplicity of this solution. Let $F_\nu(x_1, \cdots, x_{k+1})=0$ be an approximating system to the original one with exactly $n_1n_2\cdots n_k$ distinct solutions; the minimum number of these (over all approximating systems) which approach the given solution of the original system is defined as its multiplicity. The author shows that this is equivalent to the usual definition (in terms of the resultant) in case the number of solutions is finite, but that otherwise it does not give what is demanded by intuition. Severi's second definition, based on the Kronecker elimination method, is shown by an example to be inconsistent with Bézout's theorem. A proof of this theorem by Severi (also based on Kronecker's method) [Lezioni di Analisi, vol. 1, second ed., Bologna, 1941, p. 408] is stated by the author to make tacit use of certain false lemmas. A resultant criterion is given for a system of arbitrarily many equations to have an infinity of solutions.

I. S. Cohen (Cambridge, Mass.).

Purcell, Edwin J. Variety congruences of order one in n-dimensional space. Amer. J. Math. **66**, 621–635 (1944). [MF 11399]

A variety congruence of order one in n-dimensional projective space $[n]$ is an algebraic ∞^k-system of varieties V^h_{n-k} such that a generic point of $[n]$ determines one and only one V^h_{n-k} of the system. In the order h and dimension $n-k$, h and k are positive integers, $k\leqq n$. The case $h=1$ has been treated in a former paper [Trans. Amer. Math. Soc. **54**, 57–69 (1943); these Rev. **5**, 157]. The author derives the characteristics of variety congruences in $[n]$ and classifies them. When $h=2$, $n=k$, each V_0^2 of the system consists of two points P and P_1 which are pairs in a Cremona involution in $[n]$. A hitherto undiscovered Cremona involution in [3] is developed in detail to illustrate the usefulness of this new point of view. Curve congruences of order one in [3] are also discussed. Many results previously obtained by writers on Cremona transformations are special cases of this paper.

T. R. Hollcroft.

Chabauty, Claude. Démonstration de quelques lemmes de rehaussement. C. R. Acad. Sci. Paris **217**, 413–415 (1943). [MF 11681]

Let C be an algebraic curve over an algebraic numberfield. A point $P\epsilon C$ is called pseudo-integral with respect to an integer m if the denominators of its coordinates are relatively prime to m. Let $F(P)$ be the residue class field of C at P. Then an integral function f on C is called "distinguished" if the value of f at a pseudo-integral point P lies in a field $K\geqq F(P)$ such that the discriminant of $K/F(P)$ divides a constant which is independent of m. The author proves the following basic lemmas. (1) On an elliptic curve there exist for every $n\geqq 3$ two distinguished functions ω,π such that $\omega^n-\pi^n=1$. (2) If C is unicursal and has at least three distinct points at infinity then there exists a pair of distinguished functions which are related by an elliptic relation. Using these facts the author indicates that it suffices to prove Mordell's hypothesis (each curve of genus not less than 2 has only a finite number of points with

coordinates in an algebraic numberfield) for curves of genus 2. *O. F. G. Schilling* (Chicago, Ill.).

Zariski, Oscar. The theorem of Bertini on the variable singular points of a linear system of varieties. Trans. Amer. Math. Soc. 56, 130–140 (1944). [MF 10793]

The paper gives a new formulation of Bertini's theorem: a variety V_{r-1} which varies in a linear system on V_r cannot have variable singular points outside the singular locus of V_r and outside the base locus of the linear system. Given a variety V of r dimensions defined over a field k, and n indeterminates $u_1, \cdots, u_n$, a variety $V/k(u)$, called the extension of V/k, is defined over $k(u)$. Any variety W/k on V/k defines an extension $W/k(u)$ on $V/k(u)$, and a variety $W^*/k(u)$ defines a contraction W/k on V/k. If $W/k(u)$ is the extension on W/k, $W^*/k(u) \subseteq W/k(u)$. Given a linear system $|F(\lambda)|$ on V/k of dimension n, a variety F^* of $r-1$ dimensions is defined on $V/k(u)$ from the generic variety of $|F(\lambda)|$. It is proved that, if W^* is a subvariety of F^* and W is its contraction, the statement that W^* is singular on F^* implies that W is either a singular variety of V/k or a base variety of $|F(\lambda)|$. The theorem imposes no condition on the base field k. When k is of characteristic zero it is shown to imply Bertini's theorem as stated above. *W. V. D. Hodge* (Cambridge, England).

Zariski, Oscar. The compactness of the Riemann manifold of an abstract field of algebraic functions. Bull. Amer. Math. Soc. 50, 683–691 (1944). [MF 11277]

Let Σ be a field of algebraic functions of several variables over a ground field k. The totality of zero-dimensional valuations of Σ/k constitutes the Riemann manifold of Σ. If V is a projective model of Σ, H any subset of V, $N(H)$ denotes the set of valuations with center on H. Now M can be topologized by taking the sets $N(H)$, where H ranges over the algebraic subvarieties of V, to be a basis for the closed sets on M. The main result of this paper is that M is then compact.

The principal deduction from this theorem is the existence of a finite resolving system for M. The method of topologizing a variety is also used to extend the notion of the product of a finite number of lines defined over a field K to any number indexed by a set A. When A is an extension field of K, the results are used to obtain information about the homomorphisms of A on K. *W. V. D. Hodge.*

Differential Geometry

Feld, J. M. On a representation in space of groups of circle and turbine transformations in the plane. Bull. Amer. Math. Soc. 50, 930–934 (1944). [MF 11566]

Analytic methods are used to derive a continuous one to one mapping of the proper and improper turbines (special one parameter sets of oriental lineal elements) of the Moebius plane on the straight lines of projective three space. This mapping leads to isomorphisms between certain important groups of transformations in the two spaces considered. *P. Franklin* (Cambridge, Mass.).

Ladue, Mary Elizabeth. Note concerning the conformal and equilong geometries of fourth and fifth order horn sets. Amer. J. Math. 67, 155–156 (1945). [MF 11930]

The author obtains expressions for the unique equilong invariants of a horn angle of fourth order and of a horn angle of fifth order. The results are stated without proof, the methods used being identical with those employed in an earlier paper [Amer. J. Math. 65, 455–476 (1943); these Rev. 5, 12]. *A. Fialkow* (New York, N. Y.).

DeCicco, John. The pseudo-angle in space of $2n$ dimensions. Bull. Amer. Math. Soc. 51, 162–168 (1945). [MF 11833]

Generalizing results of Kasner for two complex variables and four dimensions, the author defines in terms of a particular system of Cartesian coordinates an angle between a one-dimensional and a $(2n-1)$-dimensional element in Euclidean $2n$ space, the pseudo-angle. He then shows that the transformations which preserve pseudo-angles are associated with n analytic functions of n complex variables in the same way that conformal transformations in a plane are associated with one analytic function of one complex variable. *P. Franklin* (Cambridge, Mass.).

Kasner, Edward, and DeCicco, John. Geometry of scale curves in conformal maps. Amer. J. Math. 67, 157–166 (1945). [MF 11931]

This paper is devoted mainly to proofs of results previously announced by the authors [Proc. Nat. Acad. Sci. U.S.A. 30, 162–164 (1944); these Rev. 6, 18]. *S. B. Myers* (Ann Arbor, Mich.).

Kasner, Edward, and DeCicco, John. Bi-isothermal systems. Bull. Amer. Math. Soc. 51, 169–174 (1945). [MF 11834]

The pseudo-conformal group G in Euclidean 4-space R_4 is the group of real correspondences induced among the points of R_4 by pairs of functions of two complex variables; G is characterized by the preservation of the pseudo-angle between intersecting curves and hypersurfaces. A certain class of surfaces is invariant under G, which induces a conformal correspondence between any corresponding pair of such "conformal" surfaces. A bi-isothermal system C of ∞^3 curves in R is any system pseudo-conformally equivalent to a parallel pencil of ∞^3 straight lines in R_4, and a bi-isothermal system H of ∞^1 hypersurfaces is a system pseudo-conformally equivalent to a parallel pencil of ∞^1 hyperplanes in R_4. In this paper it is shown that the pseudo-angle between a system H and a system C is a biharmonic function. It is also shown that a system H is intersected by a conformal surface in an isothermal system of curves, and that this property characterizes both systems H and conformal surfaces. *S. B. Myers* (Ann Arbor, Mich.).

Kasner, Edward, and DeCicco, John. An extension of Lie's theorem on isothermal families. Proc. Nat. Acad. Sci. U.S.A. 31, 44–50 (1945). [MF 11780]

Lie's theorem on isothermal families gives a necessary and sufficient condition that a family of curves on a surface Σ represents an isothermal family, in terms of coordinates (x, y) for which the line-element of Σ is $ds^2 = E(x, y)(dx^2+dy^2)$. In this paper a necessary and sufficient condition is obtained in terms of general coordinates (x, y).
S. B. Myers (Ann Arbor, Mich.).

DeCicco, John. Dynamical and curvature trajectories in space. Trans. Amer. Math. Soc. 57, 270–286 (1945). [MF 12134]

In this paper it is proved that the only quintuply-infinite systems of curves which are simultaneously dynamical and curvature trajectories in space are trajectories due to one

of three projectively invariant classes of fields of force: (I) those whose lines of force all lie in a pencil of planes; (II) those whose lines of force are orthogonal to a family of doubly-infinite circular helices, all of which possess the same axis and the same period; (III) those of the central or parallel type. This generalizes Kasner's earlier result for the plane case, in which only the fields of class (III) appear.
P. Franklin (Cambridge, Mass.).

Schafer, Alice T. Two singularities of space curves. Duke Math. J. **11**, 655–670 (1944). [MF 11569]

The singular points studied are inflection points and planar points. The curve is assumed to be analytic and thus can be represented in the neighborhood of the point in question by power series. Choice of the proper projective coordinate system permits the reduction of these power series to simple canonical forms. With the aid of these canonical power series, the author derives numerous theorems concerning surfaces which osculate the given curve at the singular point, concerning the sections of the tangent developable in the neighborhood of the singular point and concerning the projections of the given curve.
G. A. Hedlund (Charlottesville, Va.).

Grove, V. G. A general theory of surfaces and conjugate nets. Trans. Amer. Math. Soc. **57**, 105–122 (1945). [MF 11913]

The author has previously shown [Bull. Amer. Math. Soc. **45**, 385–398 (1939)] that certain coefficients appearing in the projective analogue of the Gauss equations of a surface behave like components of a linear connection under change of surface parameters and thus give rise to a tensor calculus. He now uses this formalism as basis for the projective study of nondevelopable surfaces and conjugate nets in terms of arbitrary surface parameters. The effect of change of the congruence of "normals" appearing in the generalized Gauss equations is investigated. It leads to the result that all congruences conjugate to a given surface may be determined without quadratures regardless of surface parametrization. They depend on one arbitrary function. Reciprocal congruences are studied briefly and the condition for a ruled surface is expressed as the vanishing of an invariant. There is a fairly extensive section on conjugate nets. Here covariant configurations such as foci, rays, axes, etc. are discussed with a new notation and several theorems immediately deduced. The effect of a change in proportionality factor of the underlying homogeneous coordinates is worked out and combined with the previously considered change of "normals" to find a canonical form for the fundamental equations together with its geometric interpretation. In a final section the fundamental equations and certain of the derived quantities are specialized to the metric case. There results a characterization of constant curvature through reciprocal congruences, and of metric normals in the class of conjugate congruences. Lastly it is shown how a metric tensor can be defined on a projective surface through a choice of ideal points on a given conjugate congruence.
J. L. Vanderslice (College Park, Md.).

Chang, Su-Cheng. On the quadrics associated with a point on a surface. Bull. Amer. Math. Soc. **50**, 926–930 (1944). [MF 11565]

In connection with a study of the canonical pencil of lines associated with a point on an analytic surface S, the author gives the following construction for the cone of Segre. Two Moutard quadrics associated with the tangents t and t' at a point of S intersect in the asymptotic tangents and a conic c. As t' moves into coincidence with t, the plane of c in its limiting position envelops the cone of Segre. It is also shown that a particular quadric in the Darboux pencil has certain properties not shared by other quadrics of that pencil.
T. R. Hollcroft (Aurora, N. Y.).

Vincensini, Paul. Sur certaines surfaces à lignes de courbure planes. Ann. Sci. École Norm. Sup. (3) **59**, 141–164 (1942). [MF 11881]

Explicit equations are derived for all two-dimensional surfaces which have the joint properties: (1) the lines of curvature are plane curves, and (2) the "mean involute" (the locus of the midpoints of the segments joining corresponding pairs of centers of principal curvature) is a plane. In addition the author derives the equations of the congruences of normals to these surfaces and the equations of the focal nappes of these congruences. The "mean involute" is generalized to an involute which is the locus of a point dividing in a constant ratio K the segment joining corresponding pairs of centers of principal curvature. For a surface satisfying condition (1) this involute is a plane only in the case of the "mean involute" ($K=-1$). The author also solves the problem of finding the surfaces which are envelopes of a one-parameter family of spheres and for which this involute is a plane. Special attention is paid to the case $K=-1$.
C. B. Allendoerfer.

Eika, T. Beitrag zur Berechnung der geodätischen Linie und der geographischen Koordinaten. Schweiz. Z. Vermessgswes. Kulturtech. **42**, 108–116, 145–147 (1944). [MF 10940]

Löbell, Frank. Zur Theorie der Flächenabbildungen. Math. Z. **49**, 427–440 (1944). [MF 11994]

The author discusses in an elementary fashion a correspondence between two surface elements in Euclidean space.
S. B. Myers (Ann Arbor, Mich.).

Garnier, René. Sur l'unicité de la surface minima inscrite dans un quadrilatère. C. R. Acad. Sci. Paris **217**, 420–421 (1943). [MF 11661]

The author again [cf. C. R. Acad. Sci. Paris **217**, 60–62 (1943); these Rev. **6**, 21] uses the theory of differential equations to establish the existence and uniqueness of a regular minimal surface bounded by a given skew quadrilateral Q, by showing that the solution $\lambda_1(t)$ of a certain differential equation has one and only one root in the interval $(0, 1)$. The result is attained by showing first that the number of roots is invariant under certain deformations of Q, and then that for a symmetric quadrilateral there is exactly one root.
E. F. Beckenbach (Austin, Tex.).

Bernstein, S. Nouvelles généralisations du théorème de Liouville et son extension aux équations du type parabolique. C. R. (Doklady) Acad. Sci. URSS (N.S.) **42**, 103–107 (1944). [MF 11626]

Let the function (*) $z=f(x, y)$ be of class C'' for all finite (x, y), and let P and L be the lower limits of p^2+q^2 at points where the curvature of the surface (*) is positive or negative, respectively. The author has shown [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **6**,

285–290 (1942); these Rev. 5, 14] that if $L<P$ then for any given $h>0$ there exists a $\delta>0$ such that

$$|z| \geqq [\delta^2(x^2+y^2)+h^2]^{\frac{1}{2}}$$

at some point on the surface; that is, we do not have $\lim z/\rho=0$ for $\rho^2=x^2+y^2\rightarrow\infty$, so that the growth of z is not less than one.

It is now shown that, if the order of growth of the surface (*) is less than one, if there is a number $P>0$ such that on the part S_P of (*) where $p^2+q^2<P$ we have

$$A(x, y, z; p, q, r, s, t)r+2B(x, y, z; p, q, r, s, t)s \\ +C(x, y, z; p, q, r, s, t)t=0$$

and (**) $AC-B^2\geqq 0$, and if the set of points of discontinuity of A, B, C, and the set of points where the sign of equality holds in (**) are nowhere dense on S_P, then the surface (*) reduces to a horizontal plane. For surfaces of parabolic type, with $AC-B^2=0$, the above general form of the theorem of Liouville holds only in a weaker form, with the horizontal plane replaced by a cylindrical surface of horizontal generatrix. *E. F. Beckenbach.*

Lense, Josef. Über einige Determinanten aus der Theorie der mehrdimensionalen Mannigfaltigkeiten. Math. Ann. 119, 216–220 (1944). [MF 11897]

For a vector in R_n (complex n-space), depending on a number of parameters, a determinant is formed, the elements of which are the inner products of the derivatives of the vector up to a certain order. Under transformation of the parameters the determinant is multiplied by a certain power of the functional determinant; this weight is determined. An application is made to isotropic vectors in the osculating space of an m-dimensional manifold in R_n.

H. Samelson (Syracuse, N. Y.).

Sen, R. N. On parallelism in Riemannian space. Bull. Calcutta Math. Soc. 36, 102–107 (1944). [MF 11843]

In a Riemannian space of n dimensions it is usual to study the parallel displacement of a vector by means of the Levi-Civita definition of parallel displacement. In this paper two types of parallel displacement are studied by the introduction of two arbitrary symmetric linear connections. Various tensors expressible in terms of these linear connections are obtained and their geometrical significance is explained. The arbitrary linear connections are then specialized in several ways by means of orthogonal ennuples and relationships between the resulting types of parallelism are given. *M. Wyman* (Edmonton, Alta.).

Wong, Yung-Chow. Quasi-orthogonal ennuple of congruences in a Riemannian space. Ann. of Math. (2) 46, 158–173 (1945). [MF 11797]

In a Riemannian space based on a positive definite quadratic tensor g_{ij}, the principal congruences of a symmetric tensor h_{ij} form an orthogonal ennuple of nonnull congruences. However, if g_{ij} is not definite, and if the elementary divisors of h_{ij} and g_{ij} are not simple, null principal congruences exist which are not linearly expressible in terms of nonnull principal congruences. This paper studies orthogonal ennuples, called quasi-orthogonal ennuples, in which some of the congruences are null and some nonnull. An invariant theory of such an ennuple is set up and applications are made to a space V_3. Geodesic null congruences receive attention; in particular, it is proved that a null congruence in V_3 is normal if and only if it is geodesic. The types of two spaces V_3 having corresponding geodesics are given, completing thereby the classification of Levi-Civita. *V. G. Grove* (East Lansing, Mich.).

Wagner, V. The inner geometry of non-linear nonholonomic manifolds. Rec. Math. [Mat. Sbornik] N.S. 13(55), 135–167 (1943). (English. Russian summary) [MF 11650]

This paper contains an extension to the nonlinear case of the concept of a nonholonomic manifold, namely, a manifold X_n of dimension n with a conical pencil of directions given at each point. The linear elements with origins in X_n and with directions belonging to the corresponding conical pencil constitute a new manifold X_n' to which a tangent plane of dimension m is attached at each point. The main aim of this paper is to establish the notion of an affine connection in X_n' and to develop its consequences. The results are then applied to the problem of Lagrange of the calculus of variations. *S. Chern* (Princeton, N. J.).

Freeman, J. G. First and second variations of the length integral in a generalized metric space. Quart. J. Math., Oxford Ser. 15, 70–83 (1944). [MF 11803]

É. Cartan's now classical development of Finsler geometry was based on a fundamental function $L(x, u)$, where x are the space coordinates and u the components of an absolute contravariant vector, the "element of support." The present paper, following a lead of Schouten and Haantjes and elaborating a paper in course of publication by E. T. Davies, extends the Cartan formalism to the case where u is of weight p. The first and second variation of the generalized length integral are written compactly in tensor form and applied as extremal conditions. The element of support introduces degrees of freedom not present in Riemannian variation problems and leads to a sequence of results depending on the type of restriction placed on the support and its variation. Results for Finsler space are of course included as the special case $p=0$. Here is a typical theorem: if the element of support is tangential to a curve C, then C is extremal for arbitrary displacement of curve and support if and only if C is autoparallel and a certain fundamental covariant vector (vanishing identically in the Riemannian case) vanishes along C. In a final section five theorems on first and second variation in the weighted Finsler space are proved and placed in juxtaposition with the corresponding known theorems [J. L. Synge] for Riemannian space. *J. L. Vanderslice* (College Park, Md.).

Lichnerowicz, André. Les espaces variationnels généralisés. C. R. Acad. Sci. Paris 217, 415–418 (1943). [MF 11716]

A generalized variational space is an n-dimensional number space with coordinates x^α in which the integral

$$J=\int_{u_0}^{u_1} H[F^u{}_{u_0}, x^\alpha(u), x'^\alpha(u)]du$$

is defined, where $F^u{}_{u_0}$ is a functional defined over the set of all differentiable curves of class at least 2 and H is homogeneous of the first degree in $x'^\alpha(u)$. Finsler spaces correspond to the case where $F^u{}_{u_0}$ does not explicitly appear in H. It is established that a Euclidean connection of linear elements can be defined in the space. *S. Chern.*

MECHANICS

***Routh, Edward John. A Treatise on Dynamics of a Particle.** G. E. Stechert and Co., New York, 1945. xi+417 pp. $5.00.

Photographic reprint of the original which was published by the Cambridge University Press, Cambridge, England, in 1898.

Hoffmann, Banesh. Kron's method of subspaces. Quart. Appl. Math. 2, 218–231 (1944). [MF 11130]

This paper contains a description, derivation and simple examples of Kron's method for finding the differential equations for a dynamical system containing workless constraints from the equations of the system without the constraints. Let the equations of the unconstrained system be (in tensor form) $f_a=m_{ab}\ddot{X}^b$. The constraints limit the motion to a subspace (Greek indices) of the unconstrained configuration space (Latin indices) $X^a=X^a(\bar{X}^\alpha)$, giving a transformation matrix $C_\alpha{}^a=\partial X^a/\partial\bar{X}^\alpha$. When the constraints are present we have

$$f_\alpha=C_\alpha{}^af_a,\quad m_{\alpha\beta}=C_\alpha{}^aC_\beta{}^bm_{ab},\quad X^\alpha=C_a{}^\alpha X^a,\quad f_\alpha=\bar{m}_{\alpha\beta}\ddot{\bar{X}}^\beta.$$

C. E. Shannon (New York, N. Y.).

Sanvisens, Francisco. Analogies between the motion of a string and the dynamics of perfect fluids. Revista Mat. Hisp.-Amer. (4) 3, 365–376 (1943). (Spanish) [MF 12179]

Kohn, Walter. The spherical gyrocompass. Quart. Appl. Math. 3, 87–88 (1945). [MF 12087]

Artobolevsky, I. I. On two new loci in the kinematics of plane mechanisms. C. R. (Doklady) Acad. Sci. URSS (N.S.) 44, 227–230 (1944). [MF 12036]

Astronomy

Liebowitz, Benjamin. A calculus of finite precision. Phys. Rev. (2) 66, 343–350 (1944). [MF 11534]

If $x_i=f_i(s, a_1, a_2)$ represents a 2-parameter family of curves in 3-dimensional space, the density of the f_i with respect to the parameters a_1, a_2 is defined by means of an expression for the normal cross-section of a sheaf of curves. This density plays the role of a probability and its derivative along a curve of the family is independent of a_1, a_2. This leads to a spatial density, independent of the parameters, which can be found as the solution of a linear second order differential equation. Applied to mechanics, the concept gives rise to an orbital density which, when divided by the velocity of the representative point, becomes the kinematical probability P. The Hamilton-Jacobi equation for $\sqrt{P}$ produces an equation for a system of n planets moving around the sun which is similar to Schrödinger's. If a nonconservative system of such moving particles is considered, there is again a governing equation of Schrödinger type, which, however, does not account for the existence of stationary states in the electron orbits in an atom. The use of complex action-functions does not remove this difficulty. *G. C. McVittie* (London).

Artemieff, N. Une méthode pour déterminer les exposants caractéristiques et son application à deux problèmes de la mécanique céleste. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 8, 61–100 (1944). (Russian. French summary) [MF 11116]

In an earlier work [same Bull. 5, 127–158 (1941); these Rev. 2, 326] the author studied critical ratios of frequencies n_0/n in the neighborhood of which the corresponding nonperturbed movements may be "nonrealizable." The possibility or impossibility of realization of a nonperturbed movement depends on whether the real parts of the characteristic exponents (c.e.'s) of a certain linear differential system are zero or not. The problem left open was to calculate the c.e.'s of certain linear differential systems with a parameter ϵ. The present work presents a substantial development of a general method for calculation of c.e.'s when the coefficients are analytic in ϵ. In the case when the second members of the linear system (in normal form) are functions of ϵ of certain types, defined for all complex ϵ, nonlocal results are obtained. Applications are made to the Mathieu equation, as well as to certain specific problems of celestial mechanics. For some problems the author's method is easier to apply than the method of Liapounoff. *W. J. Trjitzinsky* (Urbana, Ill.).

Brouwer, Dirk. Integration of the equations of general planetary theory in rectangular coordinates. Astr. J. 51, 37–43 (1944). [MF 11658]

This paper removes the necessity of introducing the perturbation of the radius vector to the disturbed planet, which Encke introduced in attempting to apply a method used by him in special perturbations to the problems of general planetary perturbations. The method is essentially that of the variation of parameters. *V. G. Grove.*

Garfinkel, Boris. On the perturbation matrices of celestial mechanics. Astr. J. 51, 44–48 (1944). [MF 11659]

The purpose of this paper is to express the Poisson and Gauss matrices, which appear in the theory of perturbations, in terms of simple constituents inherent in the theory. *V. G. Grove* (East Lansing, Mich.).

Stumpff, K. Untersuchungen über das Problem der speziellen Störungen in den rechtwinkligen Koordinaten. Astr. Nachr. 273, 105–112 (1942). [MF 11859]

The author starts with a Taylor's series development in the time for the rectangular coordinates in disturbed planetary motion. Omitting certain classes of the terms that arise in the fourth and higher derivatives, he arrives at expressions for the perturbations in the rectangular coordinates that may be computed directly, without requiring the use of a process of successive approximations. Within a single opposition of a typical asteroid the omitted terms are shown to be practically negligible. Beyond a range of from three to four months on either side of the epoch of osculation their effects increase rapidly. A separate numerical integration of the remainder terms is then necessary.

The author expects to deal further with this problem in a later article in which he hopes to arrive at rapidly convergent expressions for the rectangular coordinates in elliptic motion, upon which the practical usefulness of the method depends. *D. Brouwer* (New Haven, Conn.).

Rabe, E. Eine zweckmässige Methode zur Berechnung der allgemeinen Störungen der Planeten. Astr. Nachr. **273**, 209–229 (1943). [MF 11863]

Hansen's method of general planetary perturbations is developed with the use of an independent variable introduced by M. Brendel [Astr. Nachr. **250**, 177–208 (1933)], which corresponds to the Landen transformation in the theory of elliptic integrals. The improved convergence of the Fourier series representing the odd negative powers of the mutual distance permits a numerical development by double harmonic analysis without the introduction of the Laplacian coefficients. The procedure is treated in detail for the case of first-order perturbations produced by an undisturbed planet upon a body with negligible mass. Schedules for calculation are appended. *D. Brouwer.*

Garavito Armero, Julio. Celestial mechanics. Elliptic motion (method of Jacobi). Revista Acad. Colombiana Ci. Exact. Fis. Nat. **5**, 497–502 (1944). (Spanish) [MF 11807]

Solution of the two-body problem in spherical coordinates by Jacobi's method. *W. Kaplan* (Providence, R. I.).

García, Godofredo. Generalization of the theory of the virial. Revista Ci., Lima **46**, 281–292 (1944) = Actas Acad. Ci. Lima **7**, 225–236 (1944). (Spanish) [MF 11209]

Following a remark of G. D. Birkhoff [Dynamical Systems, Amer. Math. Soc. Colloquium Publ., vol. 9, New York, 1927, p. 292], the author generalizes several inequalities of Sundman to the case of an n-body problem for which the potential energy U satisfies an inequality which replaces the condition that U is homogeneous of degree -1. [Cf. the author's previous papers in Proc. Nat. Acad. Sci. U.S.A. **28**, 425–427, 428–432 (1942); these Rev. **4**, 57.] *W. Kaplan* (Providence, R. I.).

García, Godofredo. On the present state of the solar system. New dissipative forces acting in addition to universal gravitation. The problem of three bodies within a dissipative system. Actas Acad. Ci. Lima **7**, 351–360, 361–367 (1944). (Spanish) [MF 12092]

The first ten pages are identical with a paper already quoted from a reprint [these Rev. **6**, 75].

García, Godofredo. On the present state of the solar system. Actas Acad. Ci. Lima **7**, 409–422 (1944). (Spanish) [MF 12090]

García, Godofredo. On the present state of the solar system. New dissipative forces acting in addition to universal gravitation. The problem of three bodies within a dissipative system. Revista Ci., Lima **46**, 507–584, 2 plates (1944). (Spanish) [MF 11444]

García, Godofredo. The present state of the solar system. New dissipative and gyroscopic forces which act in addition to universal gravitation. Actas Acad. Ci. Lima **8**, 3–6 (1945). (Spanish) [MF 12060]

In the first two papers the author studies the three-body problem with gravitational attraction and in addition a resistance acting on each body, perpendicular to the plane of the three bodies and proportional to the component of velocity perpendicular to the plane. The dissipation of energy is discussed and results of Lagrange, Laplace, Poincaré and Sundman for the classical three-body problem are generalized to this case. The significance of the theory with regard to an explanation of the inclination of the planetary orbits is discussed briefly.

In the third paper an additional "gyroscopic" force is introduced for each body, proportional to the vector projection of the velocity vector on the plane of the three bodies. *W. Kaplan* (Providence, R. I.).

García, Godofredo. On a new cosmogonic theory. Revista Ci., Lima **46**, 639–671 (2 plates) (1944). (Spanish) [MF 11785]

Chandrasekhar, S. On the radiative equilibrium of a stellar atmosphere. III. Astrophys. J. **100**, 117–127 (1944). [MF 11148]

The method previously developed by the author [same J. **99**, 180–190; **100**, 76–86 (1944); these Rev. **6**, 76], which replaces integrals by sums according to Gauss's formula for numerical quadratures, is applied to the equation

$$\mu dI/d\tau = I(\tau,\mu) - (3/16)\left[(3-\mu^2)\int_{-1}^{+1} I(\tau,\mu)d\mu + (3\mu^2-1)\int_{-1}^{+1} I(\tau,\mu)\mu^2 d\mu\right].$$

This is the equation of radiative transfer incorporating a "phase function" for the particular case of Rayleigh scattering. The nth approximation to the solution is worked out and the boundary conditions for the problem are introduced. The first four approximations are given numerically in detail and tables for the corresponding laws of darkening are also provided. *G. C. McVittie.*

Hydrodynamics, Aerodynamics

Lin, C. C. On the stability of two-dimensional parallel flows. Proc. Nat. Acad. Sci. U.S.A. **30**, 316–323 (1944). [MF 11360]

The author considers the question of whether laminar flow is unstable with respect to small disturbances. Let $w(y)$ be the (given) laminar velocity profile. Oscillations given by the stream function $\psi = \varphi(y)e^{i\alpha(x-ct)}$ are superimposed on the mean velocity profile. For small oscillations the Navier-Stokes equations reduce to the linear Sommerfeld equation

$$(1)\quad (w-c)(\varphi''-\alpha^2\varphi) - w''\varphi = -(i/(\alpha R))(\varphi^{iv} - 2\alpha^2\varphi'' + \alpha^4\varphi),$$

where R denotes the Reynolds number. The stability problem then becomes an eigenvalue problem of (1). The author attacks the problem along lines similar to Heisenberg's. Two methods of solving (1) are employed. (a) Writing $\eta = (y-y_0)(\alpha R)^{1/3}$, where y_0 is determined by $w(y_0) = c$, a fundamental system of equation (1) is given by

$$\varphi = \sum_{n=0}^{\infty} \epsilon^n \chi^{(n)}(\eta), \qquad \epsilon = (\alpha R)^{-1/3},$$

and the $\chi^{(n)}$ are expressible in terms of Hankel functions of order $\frac{1}{3}$. (b) The series

$$\varphi = \sum_{n=0}^{\infty} (\alpha R)^{-n} \varphi^{(n)}(y)$$

furnishes two asymptotic solutions for large values of R.

The author investigates the analytic properties of the solutions. It is found that the eigenvalue problem can be treated by means of the asymptotic solutions (b). The asymptotic solutions fail at two points on the real axis if $c<0$; if $c>0$, the solutions hold throughout. Physically, these singular points denote the existence of two inner layers in which the influence of the viscosity is decisive, and thus $\varphi^{(0)}$ in (b) ceases to give a first approximation.

The author shows that for all laminar profiles $w(y)$ (except the Couette case) there exists an instability region similar to the one computed by Tollmien for the case of boundary layer flow along a flat plate; he improves Tollmien's computations. The critical value of R, that is, the lowest Reynolds number at which amplified oscillations exist, is explicitly given for two important cases: for plane Poiseuille flow and for boundary layer flow past a flat plate (Blasius flow). The values are $R_c=16000$ for the parabolic flow and $R_c=400$ for Blasius flow.

Approximate expressions for R_c for any laminar profile $w(y)$ are given. A discussion of the stability of profiles having an inflection point and of the physical interpretation of the instability is presented. *H. W. Liepmann.*

Landau, L. Stability of tangential discontinuities in compressible fluid. C. R. (Doklady) Acad. Sci. URSS (N.S.) **44**, 139–141 (1944). [MF 12032]

The author shows that "if the jump of velocity in (a tangential) discontinuity is large enough, so that the fluid may be looked upon as compressible, the tangential discontinuities are stable with respect to infinitesimal disturbances." He also gives estimates of the minimum velocity jump necessary for stability of the sort considered (1) in the case of an ideal gas and (2) in the case where the fluid densities on the two sides of the discontinuity are widely different. *C. C. Torrance* (Washington, D. C.).

Emmons, Howard W. Shock waves in aerodynamics. J. Aeronaut. Sci. **12**, 188–194, 216 (1945). [MF 12221]

This paper reviews some of the physical aspects of shock waves as they affect the high-velocity performance of airfoils. It is shown that the conventional frictionless, adiabatic, irrotational fluid motion can admit shocks only as an added discontinuity. Real fluids with viscosity and thermal conductivity will admit shocks as a necessary part of the complete solution of high-velocity problems. A brief discussion is given of oblique shock waves and of the necessarily rotational motion in the region behind a shock wave. A few words are included on the subject of the critical Mach number for airfoils. Finally, combustion, detonation, and condensation waves are mentioned. *Author's summary.*

Grib, A. A. Influence of the point of ignition on the parameters of an air impact wave spreading due to detonation of a gas mixture. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **8**, 273–286 (1944). (Russian. English summary) [MF 12227]

Lavrentieff, M. A. A contribution to the theory of long waves. C. R. (Doklady) Acad. Sci. URSS (N.S.) **41**, 275–277 (1943). [MF 11069]

The note deals with the steady two-dimensional motion of a heavy incompressible fluid in an infinite rectangular channel. The vertical walls of the channel are parallel to an (x, y)-plane and the cross-section of the bottom by any plane parallel to the walls is a given curve $y=y_0(x)$. The author is concerned with the problem of determining the cross-section of the free surface of the fluid. He announces without proofs a new method for constructing a solution of this problem and discusses various properties of the solution for the case where $y_0(x)$ is a periodic function. *S. E. Warschawski* (Providence, R. I.).

Kron, Gabriel. Equivalent circuits of compressible and incompressible fluid flow fields. J. Aeronaut. Sci. **12**, 221–231 (1945). [MF 12219]

Carter, G. K., and Kron, G. Numerical and network-analyzer tests of an equivalent circuit for compressible fluid flow. J. Aeronaut. Sci. **12**, 232–234 (1945). [MF 12220]

The technique is similar to that used previously for Maxwell's field equations [Proc. I.R.E. **32**, 289–299 (1944); these Rev. **6**, 55] and for the elastic field [J. Appl. Mech. **11**, A-149–A-161, A-162–A-167 (1944); J. Franklin Inst. **238**, 399–442, 443–452 (1944); these Rev. **6**, 140].

Langmann, Félix F. Solution of hydrodynamical problems by means of analogies. Bol. Fac. Ingen. Montevideo **2**, (Año 8), 509–550 (1944). (Spanish) [MF 11311]

In this paper the author reviews the methods of solution of fluid flow problems by means of well-known analogies. The analogies utilized are those of a stretched membrane under normal load and the flow of an electric current in a homogeneous conductor. Among the examples considered are: (a) the two-dimensional flow of a fluid around a cylindrical obstacle; (b) fluid flow with circulation around an airfoil; (c) filtering of a fluid through a permeable medium; (d) the motion of a compressible fluid; and (e) the flow of a lubricating oil in a bearing. The procedure employed by Hele Shaw is described and the membrane and electrical analogy illustrated for these problems. The method of Taylor utilizing electric current in a tank of variable depth for (d) is described, and an electrical network analogy for (e) is mentioned. *H. Poritsky* (Schenectady, N. Y.).

Weiss, P. On hydrodynamical images. Arbitrary irrotational flow disturbed by a sphere. Proc. Cambridge Philos. Soc. **40**, 259–261 (1944). [MF 11873]

The author considers the potential flow of an incompressible fluid caused by a sphere placed in a given irrotational flow with velocity potential $\varphi_0(x, y, z)$. If the equation of the sphere is $r=a$, where $r^2=x^2+y^2+z^2$, the disturbance potential is shown to be

$$\phi_1=\frac{a}{r}\phi_0\left(\frac{a^2x}{r},\frac{a^2y}{r},\frac{a^2z}{r}\right)-\frac{2}{ar}\int_0^a\lambda\phi_0\left(\frac{\lambda^2x}{r^2},\frac{\lambda^2y}{r^2},\frac{\lambda^2z}{r^2}\right)d\lambda.$$

Four examples of the application of this formula are given. *C. C. Lin* (Pasadena, Calif.).

Jurney, W. H. Note on flow in canals. Quart. Appl. Math. **2**, 342–346 (1945). [MF 11775]

The author solves the problem of water elevation in a semi-infinite canal of constant cross section. The canal is originally at rest, but at a certain instant water is pumped into the canal at one end with constant velocity. The elevation is calculated with the aid of Heaviside's solution for the infinite telegraph cable. The nondimensional numerical values for the water elevation are tabulated. *H. S. Tsien* (Pasadena, Calif.).

Dean, W. R. Note on the shearing motion of fluid past a projection. Proc. Cambridge Philos. Soc. 40, 214–222 (1944). [MF 11868]

In two-dimensional slow viscous fluid motion, the stream function ψ satisfies $\nabla^4\psi=0$. The author considers shear flow in the region $y>0$ and above a rigid boundary which obstructs the flow. The boundary of this region is mapped upon the unit circle in the ζ plane by

$$x+iy=z=2i\{a(1+\zeta)^{-1}+P_n(\zeta)\},$$

where P_n is a polynomial of degree n. In the z plane the stream function has the form $\psi=y^2+f(z,\bar{z})$, and

$$\psi=(1-\zeta\bar{\zeta})^2F(\zeta,\bar{\zeta})$$

in the ζ plane in order to satisfy the boundary conditions on $|\zeta|=1$. Contour integration is avoided in determining ψ and, for the cases studied, only the solution of n linear equations is required to find appropriate coefficients in F in terms of those given in $z=\omega(\zeta)$. In case the coefficients in $\omega(\zeta)$ are complex, the boundary is unsymmetrical. The author illustrates this case by an example for shear flow over an unsymmetrical boundary with a sharp projecting edge at which the fluid breaks away and forms a backwater.

D. L. Holl (Ames, Iowa).

Müller, Wilh. Zur Theorie der Kräfte bei der beschleunigten Bewegung eines Körpers in der reibungslosen Flüssigkeit. Ing.-Arch. 14, 332–350 (1944). [MF 11442]

The author gives a systematic treatment of the force system acting on a solid body moving nonuniformly in a perfect fluid. Because of the nonuniformity of the motion, it is necessary to consider the absolute velocity of the fluid. The motions studied are limited to potential flows with vanishing velocity at infinity.

General formulae are derived for the force and the moment of force acting on the body, for the energy and the impulse of the fluid motion, and for the virtual mass associated with the body. The force system acting on a turbine wheel in pure rotation is considered as an example.

Two-dimensional motions are treated in greater detail, using complex representation. As illustrative examples, the author treats the following three cases: (a) the rotation of an elliptic cylinder about an axis parallel to the axis of the cylinder, (b) the rotation of an elliptic cylinder about its center combined with a translation of its center and (c) the limiting case of a flat plate. In the last case, for uniform translation with circulation, the force is found to be normal to the direction of motion. This result is compared with the earlier results of U. Cisotti [Atti Accad. Naz. Lincei (6) 5, 16–21, 666-670 (1927)] and M. A. Omara [Philos. Mag. (7) 27, 200–211 (1939)], who found the force to be normal to the plate. [Reviewer's remark. Some of the formulae obtained by the author are very puzzling. For example, if one puts $a=b$ in equation (64) of his paper, one obtains a finite moment about the axis of symmetry of a circular cylinder moving in a perfect fluid. This is very difficult to understand, since the only forces acting on the cylinder are pressure forces passing through its axis of symmetry.]

C. C. Lin (Pasadena, Calif.).

Haskind, M. D. The oscillation of a body immersed in heavy fluid. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 8, 287–300 (1944). (Russian. English summary) [MF 12228]

The paper is concerned with the plane problem of determining the wave motion of a heavy incompressible fluid of finite depth due to the periodic oscillation of an immersed rigid body having an arbitrary form. This problem is solved by the method developed by N. L. Kotchin.

The author deduces the formulae for hydrodynamic forces and couples acting on the oscillating contour, for the amplitude of waves spreading due to oscillation and for the energy supplied to the generated waves by the contour.

As examples he considers the approximate solutions for a pulsating disk, for an oscillating disk and for a thin symmetrical profile oscillating in the direction of its axis vertically below the fluid surface.

From the author's summary.

Cowgill, Allen P. The mathematics of weir forms. Quart. Appl. Math. 2, 142–147 (1944). [MF 10807]

Following Brenke [Amer. Math. Monthly 29, 58–60 (1922)], the author formulates the problem of flow through a weir as follows. The quantity of flow of a stream of depth h is given by

$$C\int_0^h [2g(h-x)]^{\frac{1}{2}}f(x)dx,$$

where $f(x)$ is the width of the weir at height x, and g and C are constants. If this is equated to the desired function $Q(h)$ relating quantity and depth, and the equation is differentiated, the result is Abel's integral equation. The first case considered is $Q(h)=Kh^m$, K and m constants; the required function $f(x)$ for this case is determined immediately from the known solution of Abel's equation. When $m<\frac{3}{2}$, this solution gives a weir of infinite width at the base. This practical shortcoming can be corrected, for the important case $m=1$, in so-called Stout-Sutro weirs having a rectangular section at the base. The mathematical formulation of this case has been treated by Pratt [Engrg. News 72, 462–463 (1914)] by series solutions; the present author applies the same technique as for the simpler weirs described above. Finally, the same integral-equation procedure is applied to the general case in which $Q(h)$ is given by a convergent power series in h.

W. R. Sears.

Szczeniowski, B. Flow of gas through a tube of constant cross-section with heat exchange through the tube walls. Canadian J. Research. Sect. A. 23, 1–11 (1945). [MF 11798]

The author investigates the problem of flow of a compressible gas in a duct of constant cross-sectional area with heat addition or subtraction. The flow is assumed to be one-dimensional with uniform distribution across any section of the duct. The calculation is based upon the three conservation equations for mass, for momentum and for energy. Several new and important results are obtained. For instance, no heat can be added to a stream at Mach number $M=1$. Thus, the velocity of the gas cannot be raised above the sonic velocity by heating. On the other hand, if one cools a stream of Mach number one, there are two possibilities. Either the pressure will increase with decrease in velocity and eventually the flow velocity is zero with a pressure $(k-1)$ times the initial pressure (k is the ratio of specific heats); or the pressure will decrease with increase in velocity and eventually the temperature is zero with infinite Mach number. Furthermore, if a subsonic stream is heated, the gas temperature has a maximum at $M=k^{-\frac{1}{2}}$. In other words, for $k^{-\frac{1}{2}}<M<1$, the gas temperature is decreased by heating. The author draws some highly speculative conclusions on the heat conduction coefficients for this Mach number range. It is evident, however, that

the actual flow in a heated pipe is at least two-dimensional and more exact calculations have to be made before one can be certain of the validity of conclusions drawn from a simplified one-dimensional calculation. *H. S. Tsien.*

Görtler, H. Verdrängungswirkung der laminaren Grenzschichten und Druckwiderstand. Ing.-Arch. 14, 286–305 (1944). [MF 11439]

The usual boundary layer theory neglects the influence of the viscous layer adjacent to solid walls upon the outer ("free") stream flow. The author undertakes to compute this influence in first approximation, especially with respect to the form drag of solid bodies. To compute the change in pressure distribution due to the presence of the boundary layer, he considers potential flow for which the line $y=\delta^*(x, t)$ is a stream line; δ^* is the displacement thickness of the boundary layer. The pressure distribution obtained in this manner is continued to the wall by means of the relation $(1/\rho)\partial p/\partial y=u^2/R$, where R denotes the curvature of the wall. Thus

$$p(x, y, t)=p(x, \delta, t)-(\rho/R(x))\int_y^\delta u^2dy$$

for $0<y<\delta$. The resulting form drag W_D becomes

$$W_D=\int\left[p(x, \delta, t)-(\rho/R)\int_0^\delta u^2dy\right]\cos\varphi dx.$$

The procedure is applied to the drag of a body accelerated from rest in such a way that the relative velocity U is a constant multiple of t^n. This is an extension of the computations of Blasius who considered the case of an impulsive start ($n=0$). The case of a circular cylinder is discussed in particular detail. It is pointed out that the correction of the external pressure field due to the boundary layer is generally of the order δ/L, that is, of higher order than the terms of the boundary layer equation. Hence, in order to use the corrected pressure distribution for the computation of an improved boundary layer solution by an iteration process, terms up to the order δ/L must be retained from the Navier-Stokes equations. *H. W. Liepmann.*

Goldstein, S., and Young, A. D. The linear perturbation theory of compressible flow, with applications to wind-tunnel interference. Ministry of Aircraft Production, Aeronaut. Res. Committee, Rep. and Memoranda no. **1909** (6865), 1–20 (1943). [MF 11865]

The authors present three variants of the Prandtl-Glauert linear perturbation theory. The linearized equation for the velocity potential ϕ is

$$\beta^2\phi_{xx}+\phi_{yy}+\phi_{zz}=0, \tag{1}$$

where $\beta^2=1-U^2/a^2$, U is the (large) velocity component in the x direction and a is the velocity of sound in the undisturbed stream. If $\phi=Ux+f(x, y, z)$ is a known solution for the incompressible flow ($\beta=1$), the following solutions of (1) may be used for the analogous compressible flow ($\beta<1$):

$$\phi=Ux+\beta^{-1}f(x, \beta y, \beta z), \tag{I}$$

$$\phi=Ux+f(x, \beta y, \beta z), \tag{II}$$

$$\phi=Ux+f(x/\beta, y, z). \tag{III}$$

These amount to three different transformations relating the compressible and incompressible flows; in each, different features are maintained invariant between the two flows. These are discussed in some detail. The velocity due to circulation is evaluated, and the theory of three-dimensional wings is discussed; it is found that the only change in the lift distribution as β is reduced is that resulting from the increase of the section lift-curve slope. The effect of compressibility on the downwash at a tailplane is evaluated.

In the second part, the effects of compressibility on wind-tunnel interference are determined. The first cases treated are two- and three-dimensional airfoil tests, for which the corrections on incidence, drag, etc., are deduced by application of the linear-perturbation theory and by reference to the formulas for the corresponding incompressible cases [especially Glauert, same Rep. and Memoranda no. **1566** (1933)]. Some of the corrections are proportional to β^{-2}. The cases next treated are two- and three-dimensional blockage effects. Again the corrections are deduced from the incompressible analogies; they are proportional to β^{-3} and β^{-4}. Finally, the corrections to be applied for nonuniformity of flow in the test section are determined; the formulas obtained for incompressible flows by Taylor [same Rep. and Memoranda no. **1166** (1928); Proc. Roy. Soc. London. Ser. A. **120**, 260–283 (1928)] and Goldstein [same Rep. and Memoranda no. **1902** (1942)] are amended by insertion of factors β^{-1} and β^{-2} in certain terms.

W. R. Sears (Inglewood, Calif.).

Herrmann, A., und Lüddecke, W. Beitrag zum Problem des dynamischen Auftriebs von ebenen Tragflügeln. Ing.-Arch. 14, 306–310 (1944). [MF 11440]

H. Wagner obtained an expression for the force P acting on a two-dimensional flat plate in nonstationary motion in a frictionless fluid, namely,

$$P=-\rho(d/dt)\int_{C+C'}\varphi d\bar{z},$$

where φ is the potential of the motion, C the contour of the profile and C' the curve which makes the exterior of the profile a simply connected region. The authors apply Wagner's integral to the case of a moving flat plate which rotates about the quarter-chord point with an angular velocity $\omega(t)$. The solution is obtained by use of complex variables; P is taken to be the real part of a complex integral and φ is obtained with the help of Poisson's integral. The force P and the moment M about the center of the plate become

$$P=-(\pi\rho/16)d\omega/dt,\quad M=(\pi\rho/128)d\omega/dt$$

for pure rotation. If the plate also has a translation velocity v_P, the moment becomes

$$M=(\pi\rho/128)d\omega/dt-(\pi\rho/16)\omega v_P.$$

H. W. Liepmann (Pasadena, Calif.).

Glaser, Rudolf. Über die Berechnung der Koeffizienten einer in der instationären Tragflügeltheorie auftretenden unendlichen Matrix. Z. Angew. Math. Mech. 23, 279–289 (1943). [MF 11740]

In his work of 1939 on a wing flapping periodically, W. Schmiedler expressed the mean drag on the trailing vortex band in the form of an infinite bilinear Hermitian form

$$W_{\text{inst}}=(\pi\rho u^2/8)\sum_{r,s=1}^{\infty} b_{r,s}z_r\bar{z}_s,$$

u the horizontal velocity, in which the coefficients $b_{r,s}$ are expressed as series of integrals of the type

$$b_{r,s}=\sum_{k=1}^{\infty} k[c_{kr}(\mu)\bar{c}_{ks}(\mu)+\bar{c}_{kr}(\mu)c_{ks}(\mu)]\sin\mu d\mu,$$

where

$$c_{kr}(\mu)=(2/\pi)\int_0^\pi \exp\left(-i(\nu b/2u)\cot\mu\cos\phi\right) \times \sin k\phi \sin r\phi d\phi.$$

With the aid of rapidly converging series of Bessel functions which represent the quantities $b_{r,s}$, numerical computations are made of the coefficients with odd indices, the only ones which occur with a symmetrical distribution of circulation. The tables of b_{11}, b_{13}, b_{15}, b_{17}, b_{33}, b_{35}, b_{37}, b_{55}, b_{57}, b_{77} are for $\nu b/u=0$, 0.1, 0.2, 0.4, 0.6, 0.8 and give 6 decimal places.

The properties of the function

$$F_{m,n}(x)=\int_0^\infty J_m(t)J_n(t)(x^2+t^2)^{-\frac{1}{2}}dt$$

are discussed and some transformations are given for series of Bessel functions of type $\sum mJ_{m+r}(z)J_{m+s}(z)$.

H. Bateman (Pasadena, Calif.).

Maikapar, G. I. Aerodynamic calculation of an aerofoil of finite span. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 7, 463–469 (1943). (Russian. English summary) [MF 11265]

The author notices that the standard methods for solving Prandtl's integro-differential equation for the span-wise lift distribution fail when the lift is assumed to be a nonlinear function of the angle of attack or when the wing is twisted discontinuously. He transforms Prandtl's equation into an integral equation which can be solved by successive approximations even in the cases mentioned above. Numerical examples are given. *L. Bers* (Providence, R. I.).

Dorodnicyn, A. A. The influence of a fuselage on the distribution of lift along the span of an aerofoil. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 7, 233–244 (1943). (Russian. English summary) [MF 11227]

The method of this paper is based on the usual assumptions of Prandtl's wing theory supplemented by the assumption that the fuselage is a thin nonlifting body of revolution which may be replaced by a system of sources, sinks and doublets distributed along the axis of symmetry of the body. The determination of the lift distribution is reduced to two integral equations which are solved by successive approximations. The method permits taking account of the shape of the fuselage and the relative position of wing and fuselage. Numerical results are presented in the form of graphs. *L. Bers* (Providence, R. I.).

Andronow, A., et Bautin, N. Le mouvement d'un avion neutre piloté automatiquement et la théorie des transformations ponctuelles des surfaces. C. R. (Doklady) Acad. Sci. URSS (N.S.) 43, 189–193 (1944). [MF 11617]

The authors study the stability of a plane with an automatic pilot controlling the angular velocity of the rudder by means of a servo motor of constant speed. The differential equations are written as three first order equations with discontinuous right members, and the corresponding trajectories in 3-space are studied. The stability of the equilibrium point at the origin is analyzed by means of surface transformations. It is found that the equilibrium point is unstable unless the damping coefficients satisfy a certain inequality. *W. Kaplan* (Providence, R. I.).

Poliakhov, N. The minimum energy loss propeller. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1067**, 37 pp. (3 plates) (1945). [MF 11968]

Translation of Report no. 455 of the Central Aero-Hydrodynamical Institute, Moscow, 1939.

Yablokov, V. A. Oscillations of fluids in an rectangular basin rotating with a constant angle velocity. Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] 8, 157–175 (1944). (Russian. English summary) [MF 12145]

The object of the work is to complete geometrically the paper of Rayleigh; the connection between the number of amphidromical points in the basin and the length of this basin is settled, the character of cotidal lines and of the lines of the equal height of rise is determined; for the first ones the direction of their rotation around the amphidromical points is determined; for the second ones, their forms near the amphidromical points and special points are studied. *Author's summary.*

Theory of Elasticity

Ekstein, H. Free vibrations of anisotropic bodies. Phys. Rev. (2) 66, 108–118 (1944). [MF 11153]

The author's summary is as follows. "Approximate solutions for free vibrations of a finite anisotropic body are derived by a perturbation method. As an example, some extensional modes of thin crystal plates are calculated. Calculated frequencies and deformation patterns are compared with observations." *H. Poritsky.*

Panovko, J. G. Calculation of elastic systems subjected to variable loads. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 7, 375–378 (1943). (Russian. English summary) [MF 11239]

The author extends the formula

$$y(t)=\int_0^t p(\tau)\sin\phi(t-\tau)d\tau/M\phi,$$

where $y(t)$ is a generalized coordinate, M is the mass and ϕ is the frequency of free oscillations of the system, for an elastic system with one degree of freedom to elastic systems with n degrees of freedom subject to disturbing forces $p_1, \cdots, p_n$. A method is indicated for determining the "influence" functions r_{ik} in the relations

$$y_i(t)=\sum_1^n \int_0^t p_k(\tau)r_{ik}(t,\tau)d\tau.$$

An extension is given to systems with a continuous distribution of mass. *W. J. Trjitzinsky* (Urbana, Ill.).

Charrueau, André. Sur les équilibres limites plans des milieux homogènes. C. R. Acad. Sci. Paris **217**, 437–439 (1943). [MF 11664]

The author considers the mathematical theory of a statically determinate distribution of plane stress in a medium which has attained a limiting state of equilibrium under its own weight. The stress components σ_x, σ_y and τ_{xy} satisfy the usual equilibrium equations and the condition

$$(\sigma_x-\sigma_y)^2+4\tau_{xy}^2=f(m),$$

where f is a differentiable function of $m=\frac{1}{2}(\sigma_x+\sigma_y)$

$=\frac{1}{2}(\sigma_1+\sigma_2)$. If θ is the angle between the x-axis and the direction of the larger principal stress σ_1, the equilibrium equations are of hyperbolic type. The author investigates the characteristics (slip lines) when the Jacobians

$$\frac{\partial(m,\theta)}{\partial(x,y)},\quad \frac{\partial(\sigma_x,\tau_{xy})}{\partial(x,y)},\quad \frac{\partial(\sigma_y,\tau_{xy})}{\partial(x,y)},\quad \frac{\partial(\sigma_x,\sigma_y)}{\partial(x,y)}$$

vanish identically. One case includes the limiting equilibrium of Rankine. *D. L. Holl* (Ames, Iowa).

Shermann, D. I. A mixed problem of the elasticity theory. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **7**, 413–420 (1943). (Russian. English summary) [MF 11261]

Let L be an $(m+1)$-connected bounded domain whose boundary consists of $(m+1)$ closed curves L_ν, L_{m+1} being the exterior curve. Using a previously developed method [Schermann, C. R. (Doklady) Acad. Sci. URSS (N.S.) **27**, 911–916 (1940); these Rev. **2**, 270], the author determines the elastic field from the tangential component of the exterior forces and the normal component of the displacement vector on the boundary. That is, he solves the equation $\Delta\Delta u=0$ under the above boundary conditions. The problem is reduced to the determination of two analytic functions $\phi_k(z)$, $k=1, 2$, of the form

$$\phi_k(z)=\phi^{(k)}(z)-\sum_{j=1}^{m}(X_j^{(k)}+iY_j^{(k)})\log(z-a_j),$$

where $X_j^{(k)}$, $Y_j^{(k)}$ are constants and a_j are points of the "holes" bounded by L_j. Using Cauchy's formula, the $\phi_k(z)$ can be expressed in terms of two functions $\omega_k(t)$ defined on the boundary $\sum L_j$; a system of linear integral equations is derived for these functions. The author proves that the system possesses one and only one solution.

S. Bergman (Providence, R. I.).

Sneddon, Ian N. The stress distribution due to a force in the interior of a semi-infinite elastic medium. Proc. Cambridge Philos. Soc. **40**, 229–238 (1944). [MF 11870]

Melan's problem of plane strain for a single interior force normal to the boundary of a semi-infinite medium is obtained by a new method employing Fourier sine transforms. Results are then obtained for an interior line distribution of forces, and the variation of surface stresses is shown for $\frac{1}{4}\leqq\sigma\leqq\frac{1}{2}$ and for various depths of loading. Some comparisons are made with the analogous three-dimensional case of an imbedded circular loading [cf. Dean, Parsons, and Sneddon, same Proc. **40**, 5–19 (1944); these Rev. **6**, 27]. *D. L. Holl* (Ames, Iowa).

Murnaghan, F. D. The compressibility of media under extreme pressures. Proc. Nat. Acad. Sci. U.S.A. **30**, 244–247 (1944). [MF 11038]

Murnaghan, F. D. On the theory of the tension of an elastic cylinder. Proc. Nat. Acad. Sci. U.S.A. **30**, 382–384 (1944). [MF 11531]

Both papers are concerned with the possible influence of an already existing state of strain on the stress-strain relations during an additional infinitesimal deformation of an elastic body. The first paper deals with a specimen subjected to increasing hydrostatic pressure. Here the original isotropy of the material will be preserved during the straining of the material, but the bulk modulus need not be a constant and may depend on the pressure. It is shown that the assumption of a linear relation between bulk modulus and pressure leads to a formula which agrees very well with the available experimental evidence. The second paper deals with the tensile test of an elastic cylinder. It is pointed out that in this case the material need not retain its isotropy once it is strained. Formulas are developed which take account of this possible anisotropy due to strain. [Experiments with quasi-isotropic metal specimens do not seem to offer any evidence that such anisotropy occurs below the elastic limit. On the other hand, beyond this limit the straining of the material ceases to be a reversible process, and one of the author's essential assumptions is no longer fulfilled.] *W. Prager.*

Grioli, G. Sulle deformazioni elastiche dovute ad una coppia di braccio nullo. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **3**, 93–98 (1941) = Ist. Naz. Appl. Calcolo (2) no. **116**. [MF 11725]

Reissner, Eric. Note on the theorem of the symmetry of the stress tensor. J. Math. Phys. Mass. Inst. Tech. **23**, 192–194 (1944). [MF 11413]

By means of an example it is shown that the stress tensor will not be symmetrical in general if the stresses are discontinuous. *J. J. Stoker* (New York, N. Y.).

Reissner, Eric. On the theory of bending of elastic plates. J. Math. Phys. Mass. Inst. Tech. **23**, 184–191 (1944). [MF 11412]

The classical theory of thin elastic plates led to only two independent boundary conditions at the edge of the plate instead of the three conditions (the values of three stress resultants) which one would like to impose there on physical grounds. The author has developed a new theory which permits the imposition of three boundary conditions. In particular, the appropriate boundary condition at a free edge in the new theory is that the three stress resultants referred to above should vanish. The same simplifying assumptions with regard to the distribution of the stresses over the thickness of the plate are made as in the classical theory, namely that the "bending stresses" σ_x, σ_y and τ_{xy} are linear in the coordinate z. (The middle surface of the plate is the (x, y)-plane.) The differential equations are obtained from the Castigliano minimum principle, where the contributions of all stresses to the strain energy are taken into account and not just those contributions furnished by the bending stresses. The author's principal discovery is that the retention of the additional terms (which are of higher order in the thickness of the plate than the others) has the effect of yielding a system of differential equations of sixth order, so that three boundary conditions can be imposed. A typical differential equation of the system of four obtained is the following:

$$M_x-\frac{h^2}{5(1-\nu)}\left(\frac{\partial V_x}{\partial x}+\nu\frac{\partial V_y}{\partial y}\right)-\frac{h^2\nu P}{10(1-\nu)}=-D\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right),$$

in which M_x, V_x and V_y are stress resultants (over the thickness h of the plate), D and ν are the plate stiffness and Poisson ratio, w is the deflection of the middle surface and P is the normal pressure applied to the face of the plate. The corresponding equation of the classical theory results when the terms involving h^2 are neglected. Since the solutions obtained from the classical theory cannot satisfy all the boundary conditions imposed on the solutions from the

new theory, it follows that the latter solutions would not converge uniformly at the boundary as $h \to 0$ to those furnished by the classical theory. A boundary layer effect can be expected. The author treats this effect explicitly in a particular case and finds that this theory differs from the classical theory by terms which die out exponentially in the distance from the boundary. *J. J. Stoker.*

Willers, Fr. A. Die Stabilität von Kreisringplatten. Z. Angew. Math. Mech. 23, 252–258 (1943). [MF 11738]

The problem studied is the symmetrical buckling of a circular-ring plate of uniform thickness under internal stress such that a bending moment in the plane of the plate is produced. The outer edge of the ring is free and the inner edge is either free or clamped. Three methods are used for calculating the buckling value of the moment: (a) difference equation method, (b) Rayleigh's energy method and (c) power series method for the case of narrow rings. Methods (b) and (c) are more satisfactory. Complete numerical results are given for the ring with free edges.

H. S. Tsien (Pasadena, Calif.).

Morkovin, Vladimir. Effect of a small hole on the stresses in a uniformly loaded plate. Quart. Appl. Math. 2, 350–352 (1945). [MF 11778]

The author solves the problem of M. Greenspan [same Quart. 2, 60–71 (1944); these Rev. 5, 250] of an infinite plate under generalized plane stress, subjected to uniform loads at the infinite boundary and free from applied forces at the edge of an interior curvilinear notch or ovaloid. The stress function is determined by the method introduced by N. I. Muschelišvili. Two analytic functions $\varphi(z)$ and $\psi(z)$ are found which produce the known stresses at infinity and which satisfy

$$\varphi(z)+z\bar{\varphi}'(\bar{z})+\bar{\psi}(\bar{z})=\varphi(\zeta)+\bar{\psi}(\bar{\zeta})+\omega(\zeta)\bar{\varphi}'(\bar{\zeta})/\bar{\omega}'(\bar{\zeta})=0$$

at the edge $|\zeta|=1$ of the unit circle, upon which the curvilinear notch is mapped by the rational function

$$z=\omega(\zeta)=s\zeta+t/\zeta+r/\zeta^3, \quad r, s, t=\text{constant}.$$

When there are no unbalanced forces on the notch, the stresses are bounded at infinity, there are no singular load points, and the mapping function is rational, the analytic functions become power series in ζ and the method is much more direct than that employed by Greenspan for the same problem. [Reviewer's note. References in a paper by S. G. Lekhnitsky [C. R. (Doklady) Acad. Sci. URSS (N.S.) 31, 845–848 (1941); these Rev. 3, 95] indicate that several Russian workers have solved similar examples of isotropic and anisotropic infinite plates with "approximate square notches" by analytic functions and rational mapping functions in cases of tension, shear and flexure.] *D. L. Holl.*

Galin, L. A. Concerning the hypothesis of Zimmermann-Winkler. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 7, 293–300 (1943). (Russian. English summary) [MF 11230]

The author considers an absolutely rigid beam of finite length $2a$ and rectangular cross-section exerting pressure on a supporting elastic semi-space $z<0$. He investigates the hypothesis of Zimmermann-Winkler, according to which the displacement $w(x)$ of the beam is proportional to the average pressure $p(x)$ per unit length. The width 2δ of the cross-section is assumed to be small in comparison with the length $2a$. The pressure of the beam is assumed to be

$$p^*(\xi, y)=p(\xi)/\pi(\delta^2-y^2)^{\frac{1}{2}},$$

where $p(\xi)$ is the average pressure. The displacement of the medium in the z direction is given by a harmonic function $W(x, y, z)$ such that (1) $W(x, 0, 0)=w(x)$ for $-a<x<a$; (2) $\partial W^-/\partial z=cp(\xi)/(\delta^2-y^2)^{\frac{1}{2}}$ for $z=0$ underneath the beam and $\partial W^-/\partial z=0$ otherwise; (3) $W\sim c/(x^2+y^2+z^2)^{\frac{1}{2}}$ as $z\to\infty$, $(x^2+y^2+z^2)^{\frac{1}{2}}\to\infty$. Under these assumptions the author obtains

$$(*) \qquad w(x)=\int_{-a}^{a} p(\xi)H(x-\xi)d\xi,$$

where H is a known kernel. Assuming that w and p are proportional, (*) is a singular integral equation with unknown constant (embedding constant). A rather complicated mathematical investigation of the integral equation confirms the validity of the Zimmermann-Winkler hypothesis for all points except perhaps the angular points, where the pressure may become infinite. An analytic expression for the embedding constant $k=[p(x)/w(x)]$ is derived.

S. Bergman (Providence, R. I.).

Opatowski, I. Cantilever beams of uniform strength. Quart. Appl. Math. 3, 76–81 (1945). [MF 12252]

Cantilevers of nonuniform cross section are considered. They are loaded by their own weight and by an isolated force at the free end. Every cross section has a vertical axis of symmetry, and the line of centroids is straight. Rectangular Cartesian coordinates (U, V) are taken in the plane of the cross section, with the origin at the centroid and the V-axis directed vertically upward. Cross sections of the following type are considered: $U=u(x)u_1(t)$, $V=v(x)v_1(t)$, where x is the distance of the cross section from the free end, and t is a parameter. The functions $u_1(t)$ and $v_1(t)$ specify the type of cross section, while the functions $u(x)$ and $v(x)$ specify the variation of this cross section along the beam. The well-known equations of equilibrium of a beam are introduced, and the condition is imposed that the maximum stress be the same in all cross sections. Among others, the following geometrical properties of the cross section are assigned: (a) v is constant; (b) v is linear in x; (c) u is a constant multiple of v^n. In cases (a) and (b), u is determined in terms of hyperbolic functions and Bessel functions, respectively; in case (c), elliptic functions are involved. *G. E. Hay* (Providence, R. I.).

Krall, G. Infestonamento della lastra collaborante. Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 4, 75 pp. (1943) = Ist. Naz. Appl. Calcolo (2) no. 163. [MF 11726]

This is a detailed investigation of some aspects of the problem of the effective width of the cover sheets of wide flanged T-beams [see, for instance, S. Timoshenko, Theory of Elasticity, McGraw-Hill, New York, 1934]. The author's main concern is a hitherto disregarded nonlinear effect of sheet deflection on the strength of the beam. The author shows that this effect may be determined explicitly, by the use of the equations for finite deflections of thin plates, when the beam load consists of a uniform bending moment $M=M_0$. For more general load distributions,

$$M=\sum M_n \cos k_n x,$$

an approximation procedure is suggested which makes use of the known results of the linear theory and of the author's results for the case $M=M_0$. *E. Reissner.*

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